Effective Fractions Instruction and Interventions for Students with Mathematics Difficulties

KSMTSS & Alignment

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We will use PollEverywhere for this meeting.
To prep:
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-or-
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Agenda

- Identify common errors with fractions
- Fractions as a predictor
- Develop fraction magnitudes
- Compute fractions
- Interventions
- Connections across rational numbers

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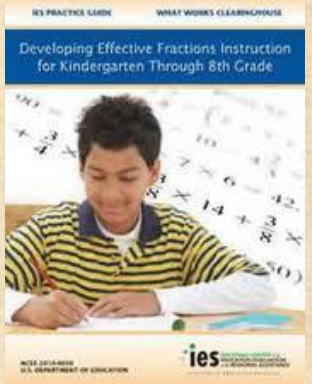
IES Practice Guide on Fractions (Siegler et al., 2010)



http://ies.ed.gov/ncee/wwc/pdf/practiceguides/fractions_pg_093010.pdf

Recommendation 1. Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts

Recommendation 2. Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward

Recommendation 3. Help students understand why procedures for computations with fractions make sense

Recommendation 4. Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems

Recommendation 5. Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them

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Students' Difficulties with Fractions

Most 8th graders did not reason that each fraction is close to 1

$$\frac{12}{13} + \frac{7}{8} = \frac{19}{21}$$

(Carpenter et al, 1980)

$$6+0.32 = 0.38$$

(Hiebert & Wearne, 1985)

43% of students don't identify the place value of the addends

$$87\% \text{ of } 10 > 10$$

(Gay & Aichele, 1997)

More than ½ of MS students

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Errors with Fractions

$3\frac{1}{3} =$

Select all that apply:

A. $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ 52%

B. $\frac{4}{4}+\frac{5}{5}+\frac{6}{6}+\frac{1}{3}$ *

C. $\frac{4}{3}$ 30%

D. $\frac{10}{3}$ *

E. $\frac{3}{3}+\frac{3}{3}+\frac{3}{3}+\frac{1}{3}$ *

Errors with Fractions survey *PollEverywhere*

*Only 2 out of 46 indicated all correct answer choices

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Common Difficulties with Fractions

(Riccomini, Hughes, Morano, Hwang, & Witzel, 2015)

Fraction Item Error Analysis of Low, Middle, and High Achieving Groups										
	Achievement Groups									
	Low				Middle				High	
Item Categories	M	SD	Rank		M	SD	Rank		M	SD	Rank
Division	.03	.15	1		.16	.31	1		.64	.44	2
Ordering	.04	.23	2		.25	.44	2		.55	.50	1
Multiplication	.51	.39	8		.42	.40	3		.88	.23	5
Word Problems	.09	.15	5		.5	.33	4		.87	.20	4
Subtraction D(1)	.08	.26	4		.62	.43	6		.91	.24	6
Subtraction S(4)	.67	.47	9		.95	.22	9		.98	.13	8
Addition S(4)	.73	.37	10		.97	.15	10		.99	.08	10
Addition D(1)	.04	.17	3		.51	.42	5		.82	.30	3
Note. (1) Different denominator, (2) Least form, (3) Equivalent form, (4) Same denominator. Rank numbers from 1-10 signify greatest frequency of errors '1' to smallest frequency of errors '10'.										

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Fractions Research

National Math Panel (2008)

- "Difficulty with fractions (including decimals and percents) is pervasive and is a major obstacle to further progress in mathematics, including algebra“
- Tom Loveless stated, "Students don't know how to translate fractions into decimals or into percentages and they can't locate fractions on a number line“

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Connect fractions to algebra

- “elementary school students’ knowledge of fractions and of division uniquely predicts those students’ knowledge of algebra and overall mathematics achievement in high school, 5 or 6 years later
- even after statistically controlling for other types of mathematical knowledge, general intellectual ability, working memory, and family income and education”
- (Siegler et al. 2012, p. 691)

Simplify this fraction $\frac{3x + 5}{3}$

Common error

$\frac{3x + 5}{3} = 1x + 5$

Consider teaching

$$\frac{3x+5}{3} = \frac{3x}{3} + \frac{5}{3} = 1x + \frac{5}{3}$$

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Sense-making: Converting fractions

Mindless Steps

Convert this mixed fraction into an "improper" fraction.

$4\ ^{2}/_{5}$

How did you do it?

Did you…

a. 4x5

b. + 2 = 22

c. $^{22}/_{5}$

Why?

Connected Procedures and Concepts

Convert this mixed fraction into an "improper" fraction.

$4\ ^{2}/_{5}$

Say, "Four and two – fifths"

$$^{4}/_{1} + {}^{2}/_{5} = {}^{20}/_{5} + {}^{2}/_{5} = {}^{22}/_{5}$$

B

Your Turn – Convert these fractions

a) $1\frac{4}{5} =$

b) $3\frac{1}{4} =$

c) $2\frac{3}{5} =$

d) $6\frac{1}{3} =$

e) $4\frac{5}{4} =$

f) $3\frac{5}{2} =$

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Sense-making: Division of Fractions

- Why is it that when you divide fractions, the answer is larger? Also, why do you invert and multiply?

$^{2}/_{3}$ divided by $^{1}/_{4} = {}^{2}/_{3}\,(^{4}/_{1}) = {}^{8}/_{3}$

$$\frac{{}^{2}/_{3}\,(^{4}/_{1})}{{}^{1}/_{4}\,(^{4}/_{1})} = \frac{{}^{8}/_{3}}{{}^{4}/_{4}} = \frac{{}^{8}/_{3}}{{}^{1}/_{1}} = {}^{8}/_{3}$$

B

Your Turn – Compute these division problems

a) $\frac{4}{5} \div \frac{1}{5} =$

b) $\frac{1}{3} \div \frac{2}{3} =$

c) $\frac{1}{4} \div \frac{1}{2} =$

d) $\frac{1}{6} \div \frac{3}{2} =$

e) $\frac{4}{3} \div \frac{1}{5} =$

f) $\frac{1}{2} \div \frac{2}{3} =$

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Why teach the basics correctly

Adding with unlike denominators

Division of fractions

$$\frac{5+\frac{1}{y}}{3+\frac{2}{y^2}} = \frac{\left(\frac{5y}{y}+\frac{1}{y}\right)}{\left(\frac{3y^2}{y^2}+\frac{2}{y^2}\right)} = \frac{\left(\frac{5y+1}{y}\right)}{\left(\frac{3y^2+2}{y^2}\right)}$$

$$= \frac{\left(\frac{5y+1}{y_1}\right)\left(\frac{y^2}{3y^2+2}\right)}{\left(\frac{3y^2+2}{y^2_1}\right)\left(\frac{y^2_1}{3y^2+2_1}\right)} = \frac{\frac{y(5y+1)}{(3y^2+2)}}{1}$$

$$= \frac{5y^2+y}{3y^2+2}$$

Why is it beneficial to understand the reasoning behind fractions computation? *PollEverywhere*

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Core Instruction with Fractions

"Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward."

(Siegler et al, 2010)

- Fraction Length-based models
- Number Line computational practice
- Area to Number line connections

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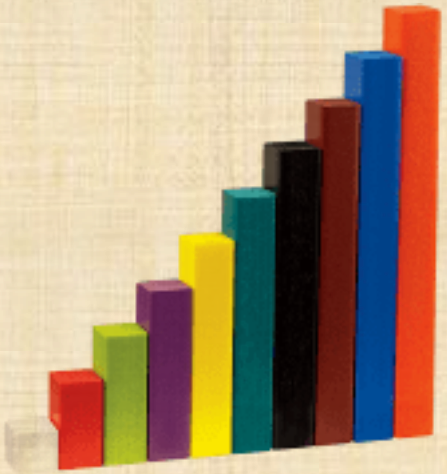
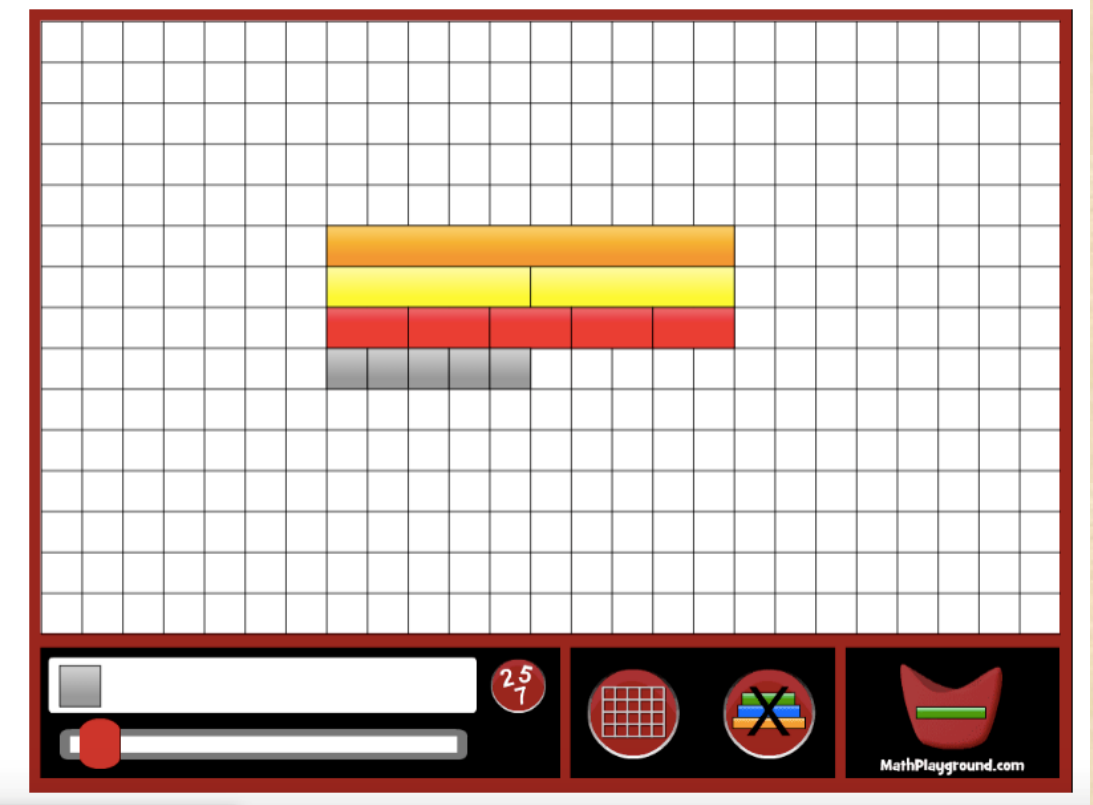
Virtual Cuisenaire Rods

https://www.mathplayground.com/mathbars.html





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Fraction Highlights within Standards

3rd grade

- Understand a fraction as a number on the number line; represent fractions on a number of them is line diagram.
- Explain equivalence of fractions, and compare fractions by reasoning about their size (it is a mathematical convention that when comparing fractions, the whole is the same size).

4th grade

- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
    - Decomposition
    - Mixed numbers with like denominators
    - Word problems
- Understand decimal notation for fractions, and compare decimal fractions.
- Limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100

5th grade

- Use equivalent fractions as a strategy to add and subtract fractions. With unlike denominators
- Interpret fractions as division of a numerator by a denominator
- Partitioning and rescaling
- Application and word problem solving

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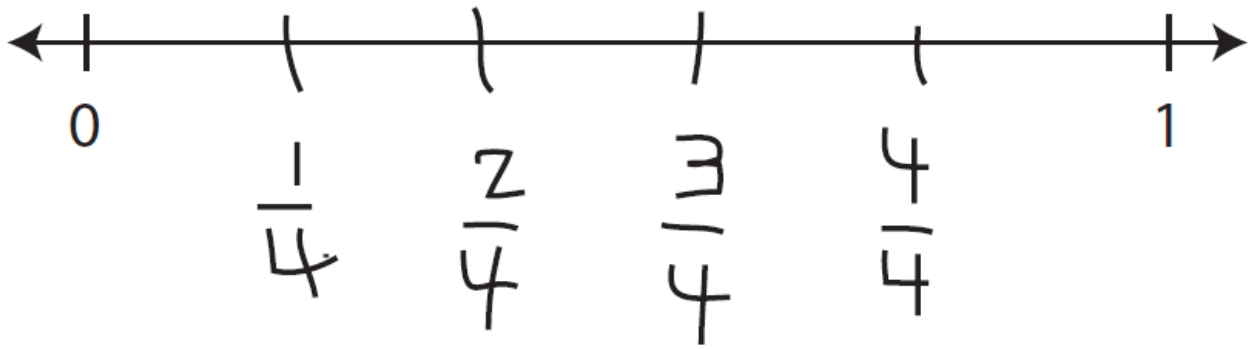
Research in the Area of Fractions and Its Application to Classroom Practices

(Beckman, 2012)

Fractions Practice Guide, Roadblock 2.1

A common misconception:
students count tick marks instead of attending to length.



The student put 4 tick marks inside the interval instead of dividing the interval into 4 equal parts.

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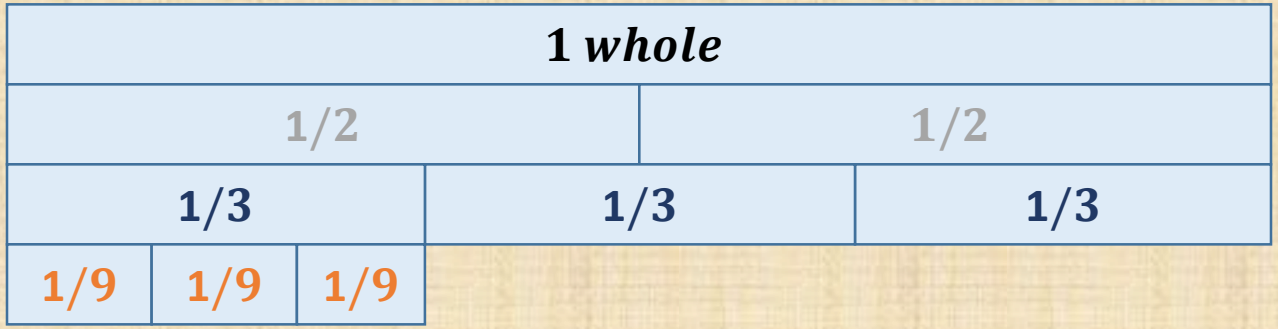
Fun with Fractions:
Making and Investigating Fraction Strips



- Students cut, fold, and color strips of paper to create length-based models of fraction lines.
- Strips are stacked in order to make comparisons
- Ask questions such as, “Which strip is one-third of the whole?” and “Which strip is one-third of one-third?”

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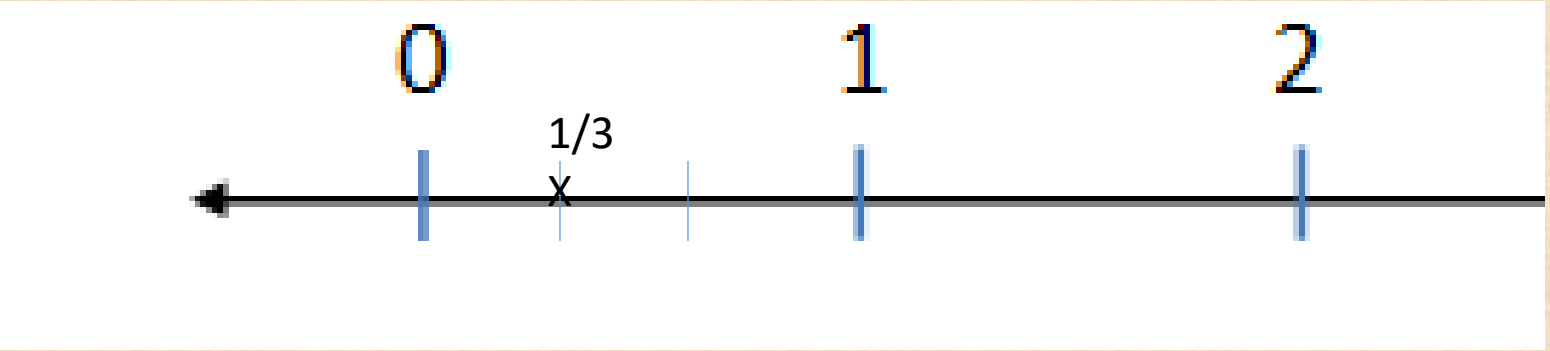
Progressions:
Identify 1/3 on a number line

Think aloud.



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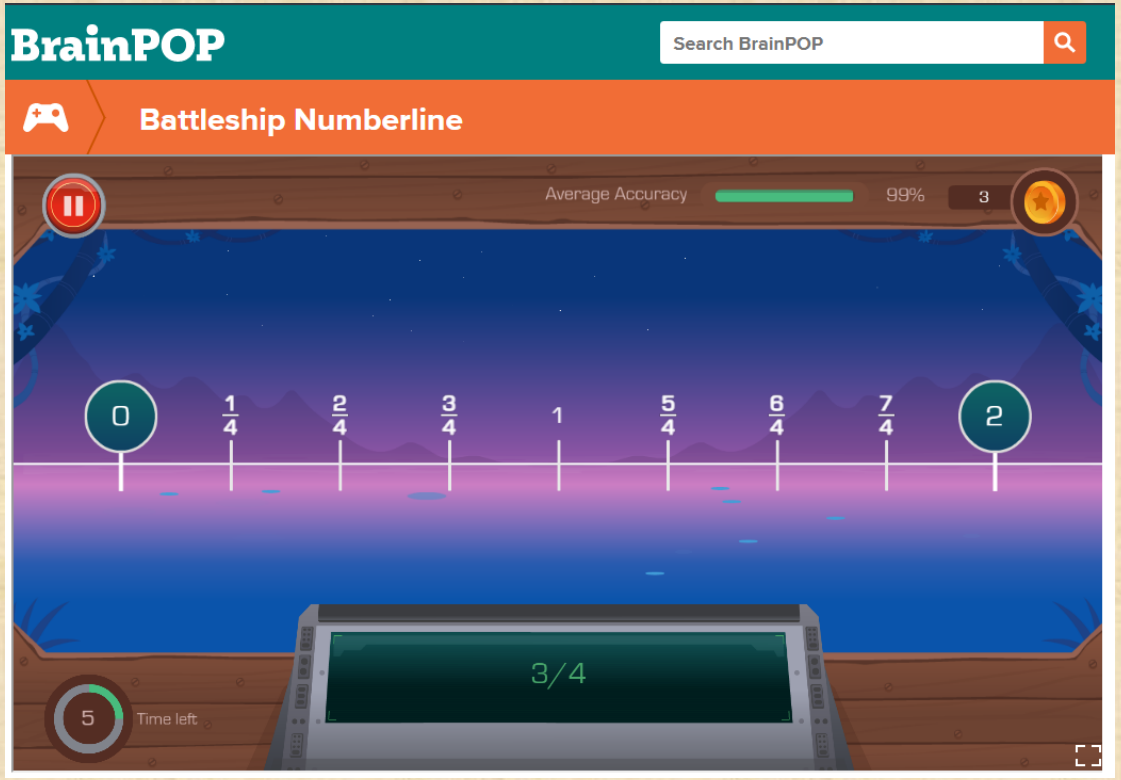
Magnitude estimates

- https://www.brainpop.com/games/battleshipnumberline/



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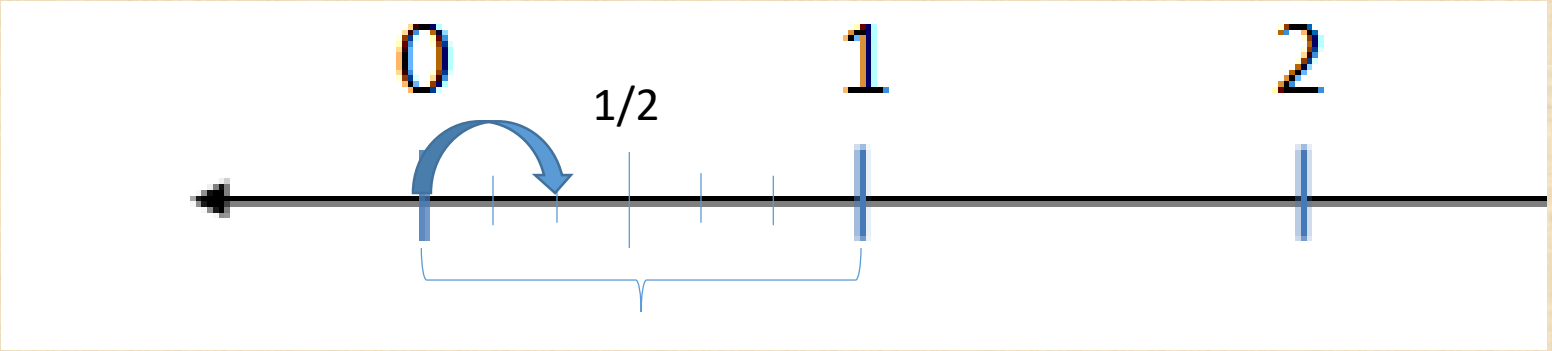
Progressions:

$(^2/_3)(^1/_2)$

Verbally say, “2/3rds of 1/2”



(2/3) (1/2) = 2/6

How could this be translated to 1/3?

(Witzel & Little)
Elementary Math

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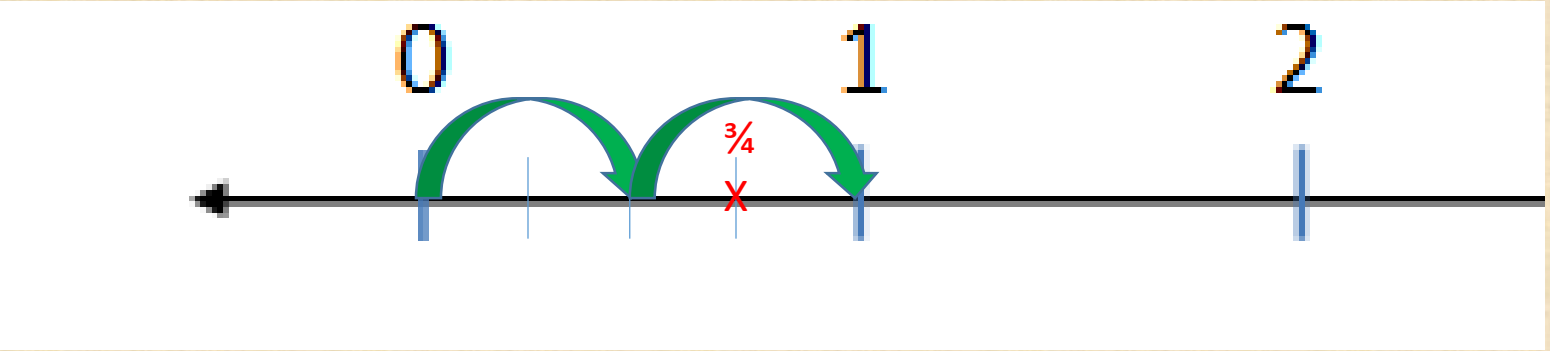
Progressions:

¾ ÷ ½

Verbally say, “How many one-halves go into three-fourths?” What is the logical answer?



Answer: $1+ {}^{1}/_{2} = {}^{3}/_{2}$

Check: $1({}^{1}/_{2}) + {}^{1}/_{2}({}^{1}/_{2}) = {}^{3}/_{4}$

(Witzel & Little)
Elementary Math

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Stages of computational proficiency development

A. Understanding key concepts

Teach students to recognize and explain operational symbols

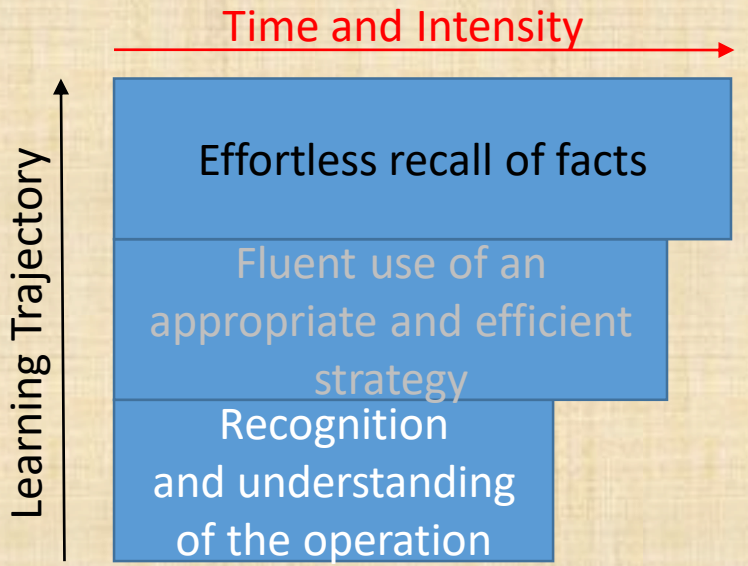
B. Fluent use of strategies

Teach efficient and appropriate computational strategies

C. Recall / Automaticity of facts

Practice incremental recall of facts

This is for fractions, too!



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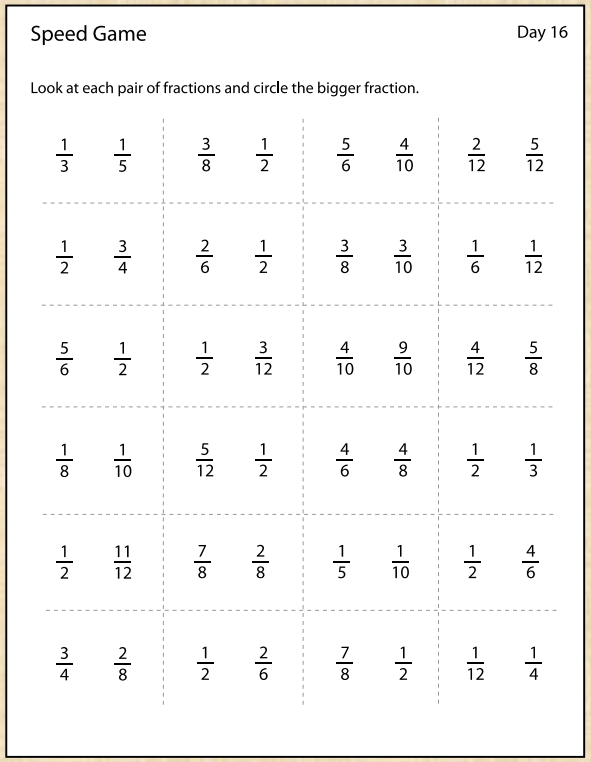
Fluency-Building Activities (Fuchs & Schumaker)

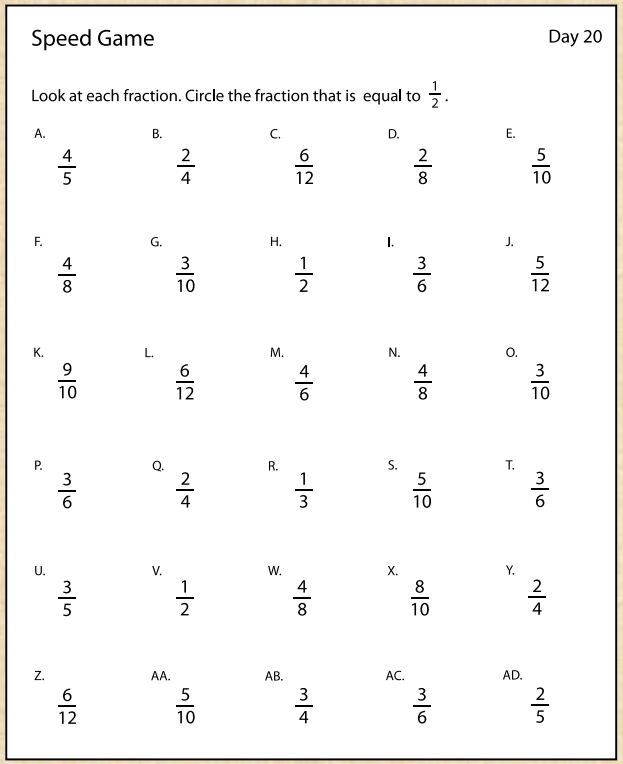
- Speed Game
  - Timed 1-minute worksheet, e.g.
    - Circle the bigger fraction (sample left)
    - Circle all fractions equal to ½ (sample right)
    - Solve all addition problems (cross out subtraction)



Speed Game — Day 16

Look at each pair of fractions and circle the bigger fraction.

$\frac{1}{3}$	$\frac{1}{5}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{4}{10}$	$\frac{2}{12}$	$\frac{5}{12}$
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{6}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{10}$	$\frac{1}{6}$	$\frac{1}{12}$
$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{12}$	$\frac{4}{10}$	$\frac{9}{10}$	$\frac{4}{12}$	$\frac{5}{8}$
$\frac{1}{8}$	$\frac{1}{10}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{4}{6}$	$\frac{4}{8}$	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{1}{2}$	$\frac{11}{12}$	$\frac{7}{8}$	$\frac{2}{8}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{4}{6}$
$\frac{3}{4}$	$\frac{2}{8}$	$\frac{1}{2}$	$\frac{2}{6}$	$\frac{7}{8}$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{4}$



Speed Game — Day 20

Look at each fraction. Circle the fraction that is equal to $\frac{1}{2}$.

A. $\frac{4}{5}$	B. $\frac{2}{4}$	C. $\frac{6}{12}$	D. $\frac{2}{8}$	E. $\frac{5}{10}$
F. $\frac{4}{8}$	G. $\frac{3}{10}$	H. $\frac{1}{2}$	I. $\frac{3}{6}$	J. $\frac{5}{12}$
K. $\frac{9}{10}$	L. $\frac{6}{12}$	M. $\frac{4}{6}$	N. $\frac{4}{8}$	O. $\frac{3}{10}$
P. $\frac{3}{6}$	Q. $\frac{2}{4}$	R. $\frac{1}{3}$	S. $\frac{5}{10}$	T. $\frac{3}{6}$
U. $\frac{3}{5}$	V. $\frac{1}{2}$	W. $\frac{4}{8}$	X. $\frac{8}{10}$	Y. $\frac{2}{4}$
Z. $\frac{6}{12}$	AA. $\frac{5}{10}$	AB. $\frac{3}{4}$	AC. $\frac{3}{6}$	AD. $\frac{2}{5}$

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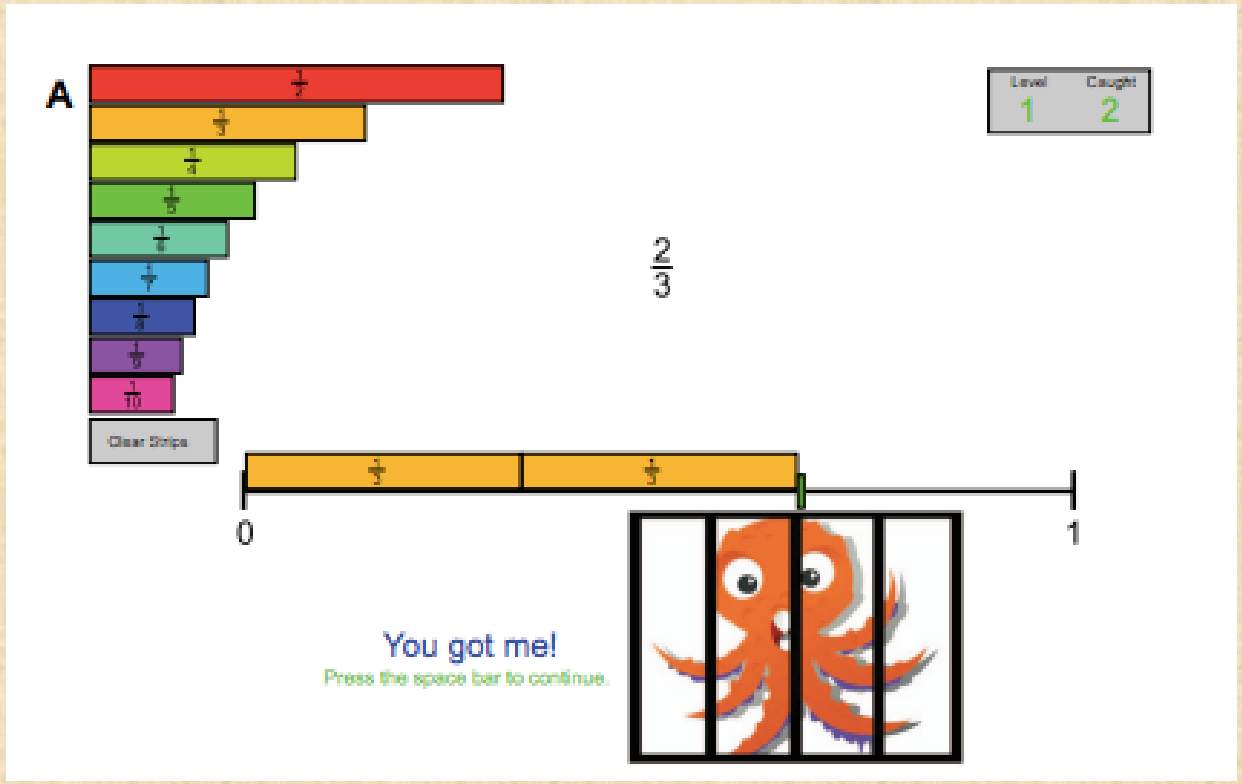
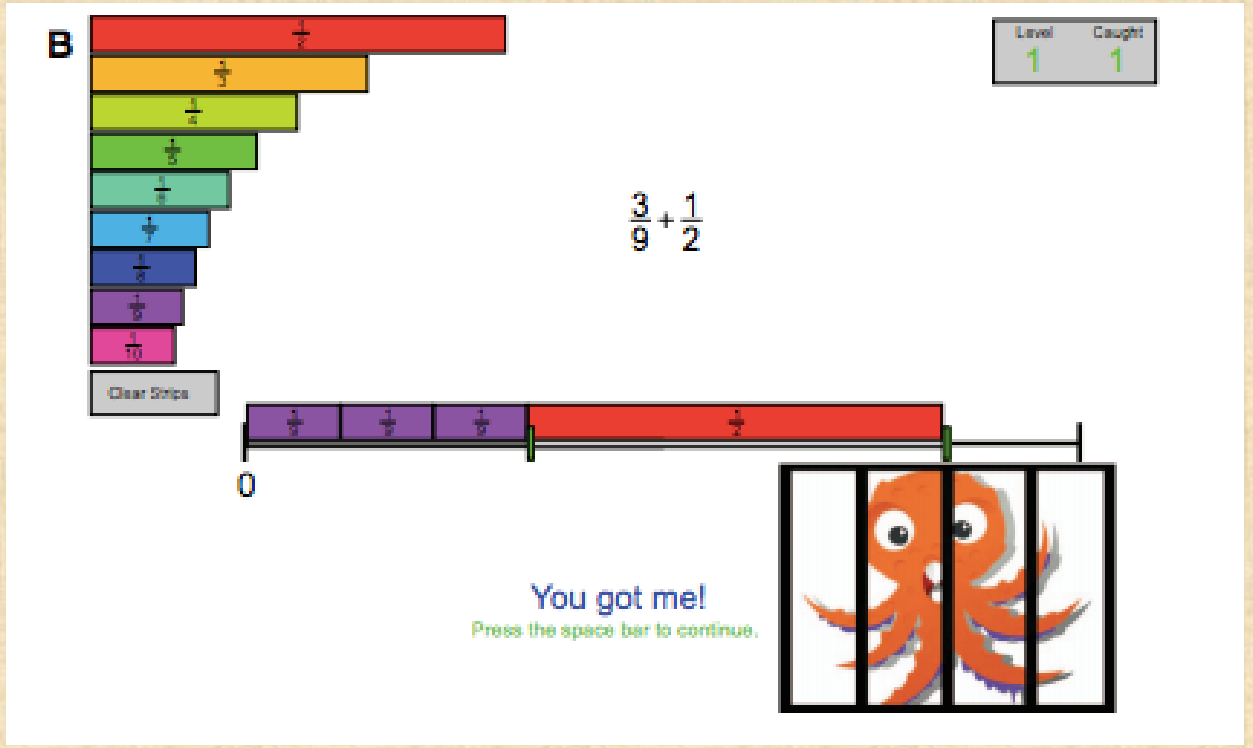
Blending Estimation with Computation (Braithwaite & Siegler, 2020)





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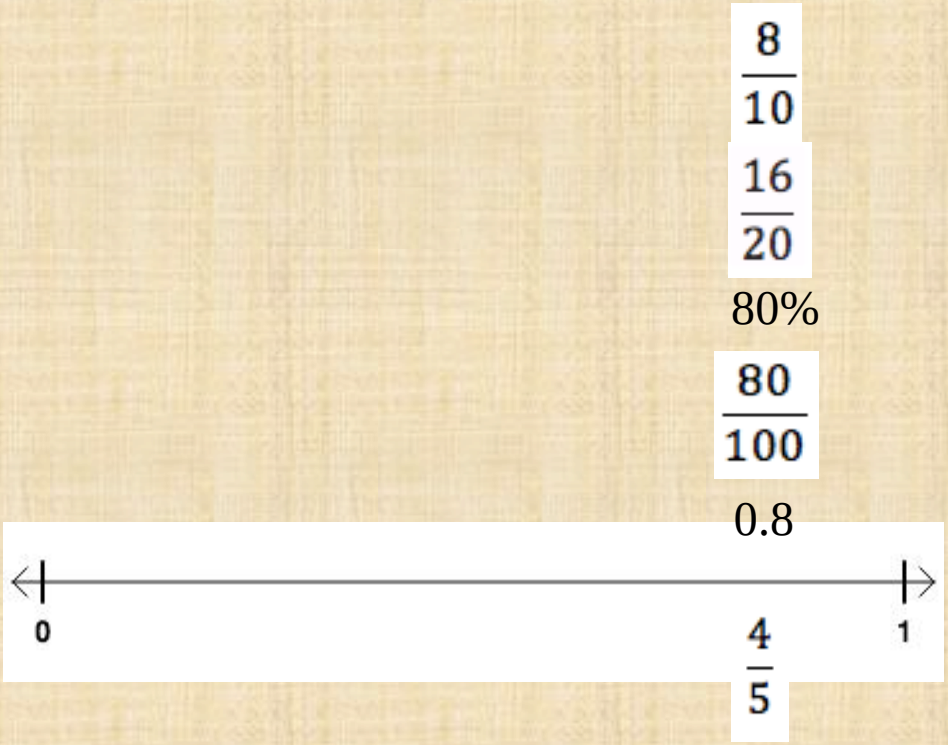
A new frontier: Relations Among Notations (Schiller, 2020)

25% vs. 2/5



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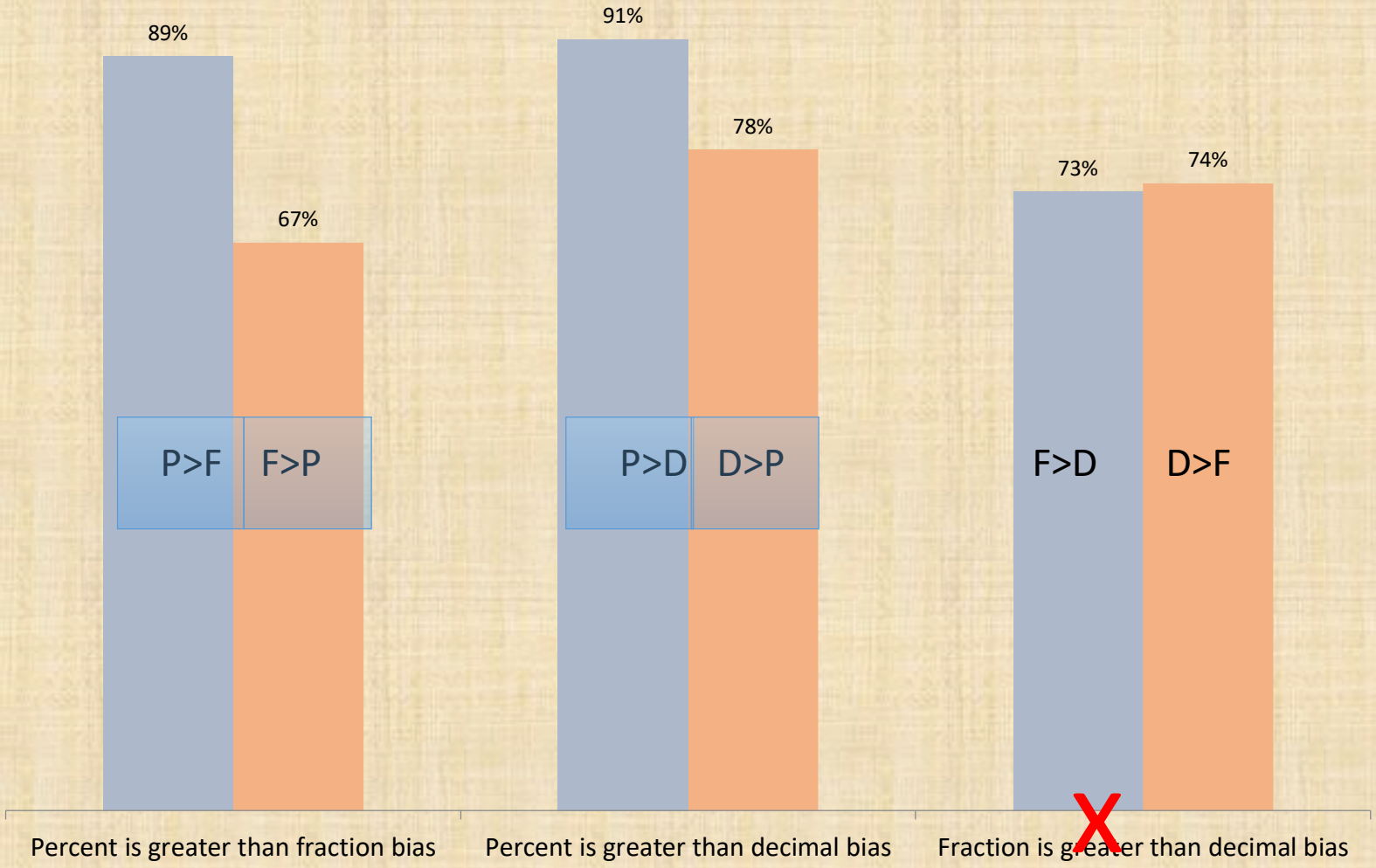
Percentages-are-larger Bias (Schiller, 2020)



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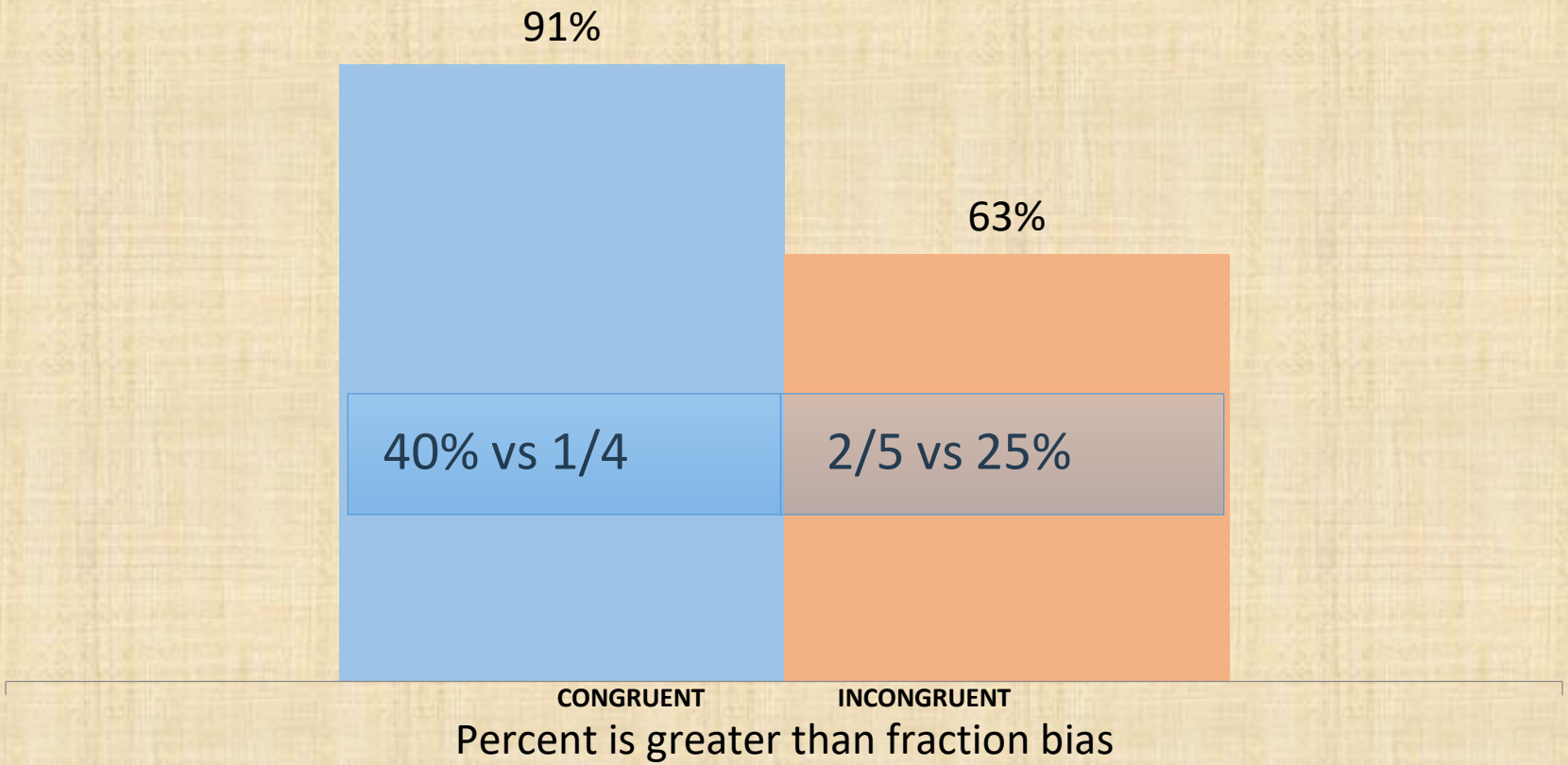
Example: Percent>Fraction Bias



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Individual differences in understanding of the relations among fractions, decimals, and percentages predict estimation ability and achievement scores (Schiller, 2020)

$$\frac{12}{13} + \frac{7}{8} \sim 2$$

B

Intensive Interventions with Fractions

- Start with the Measurement Model
- Concrete to Representation to Abstract sequence of instruction
- Aim for Abstract reasoning and proactive
- Explicit instruction
- Embed whole number Reteaching
- Continue practice at all levels, particularly abstract

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Understanding Using Multiple Representations

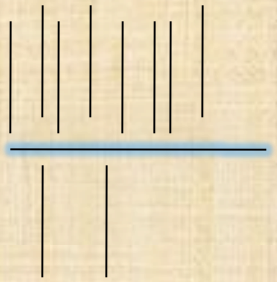
- Three stages of learning
  - C = Learning through concrete hands-on manipulative objects
  - V = Learning through visual forms of the math skill
  - A = Learning through work with abstract (Arabic) notation
- Aimed at teaching efficient and effective procedural fluency
- See IRIS Modules (https://iris.peabody.vanderbilt.edu/module/math/cresource/q2/p05/#content)

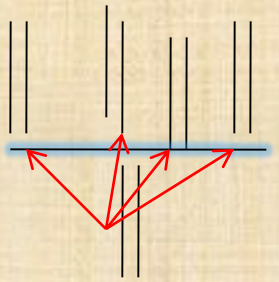
  or request videos from Dr. Witzel

B

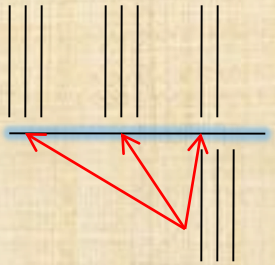
Mixed to improper and back: concrete

- 8 ÷ 2

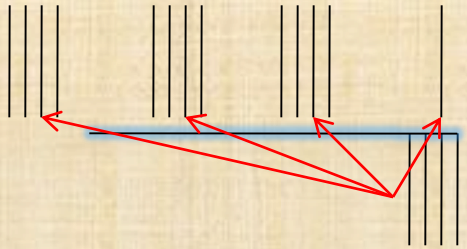




- 8 ÷ 3



- 13 ÷ 4



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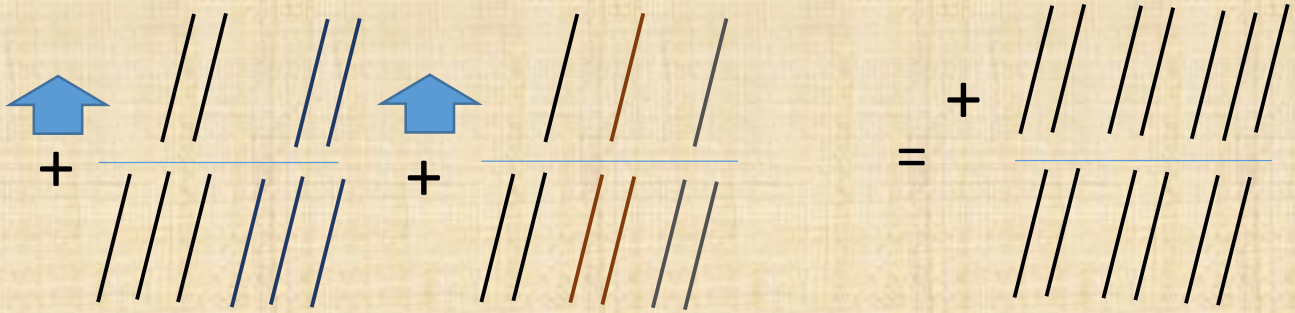
Intervention with Fractions procedures

(Watt & Therrien, 2016; Witzel & Riccomini, 2009)

$^{2}/_{3} + {}^{1}/_{2}$



$$\frac{(2+2)}{(3+3)} + \frac{(1+1+1)}{(2+2+2)} = \frac{7}{6}$$

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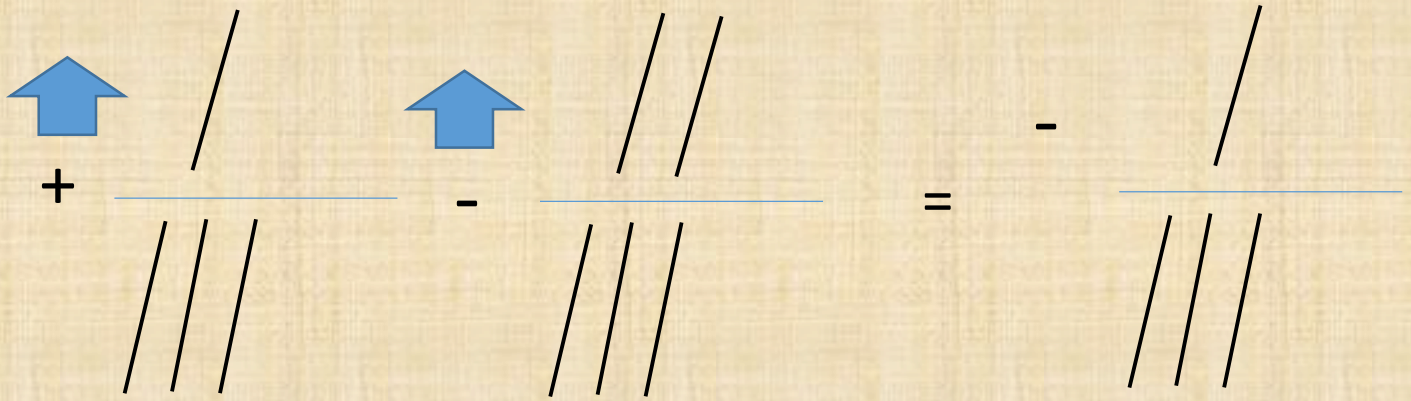
Aim interventions at procedural processes

(Watt & Therrien, 2016; Witzel & Riccomini, 2009)

$^{1}/_{3} - {}^{2}/_{3}$



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Your Turn – Compute these problems

a) $\frac{4}{5} + \frac{1}{5} =$

b) $\frac{1}{3} + \frac{2}{3} =$

c) $\frac{1}{4} + \frac{1}{2} =$

d) $\frac{5}{6} - \frac{1}{2} =$

e) $\frac{4}{3} - \frac{1}{3} =$

f) $\frac{1}{2} - \frac{2}{3} =$

L

3-2-1 Take Home

Take Aways: Name at least one thing that you can implement from this webinar *PollEverywhere*

3 things you learned

2 things you can implement with ease

1 question you still have

For more information or future collaborations, contact

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