Building Computational Fluency

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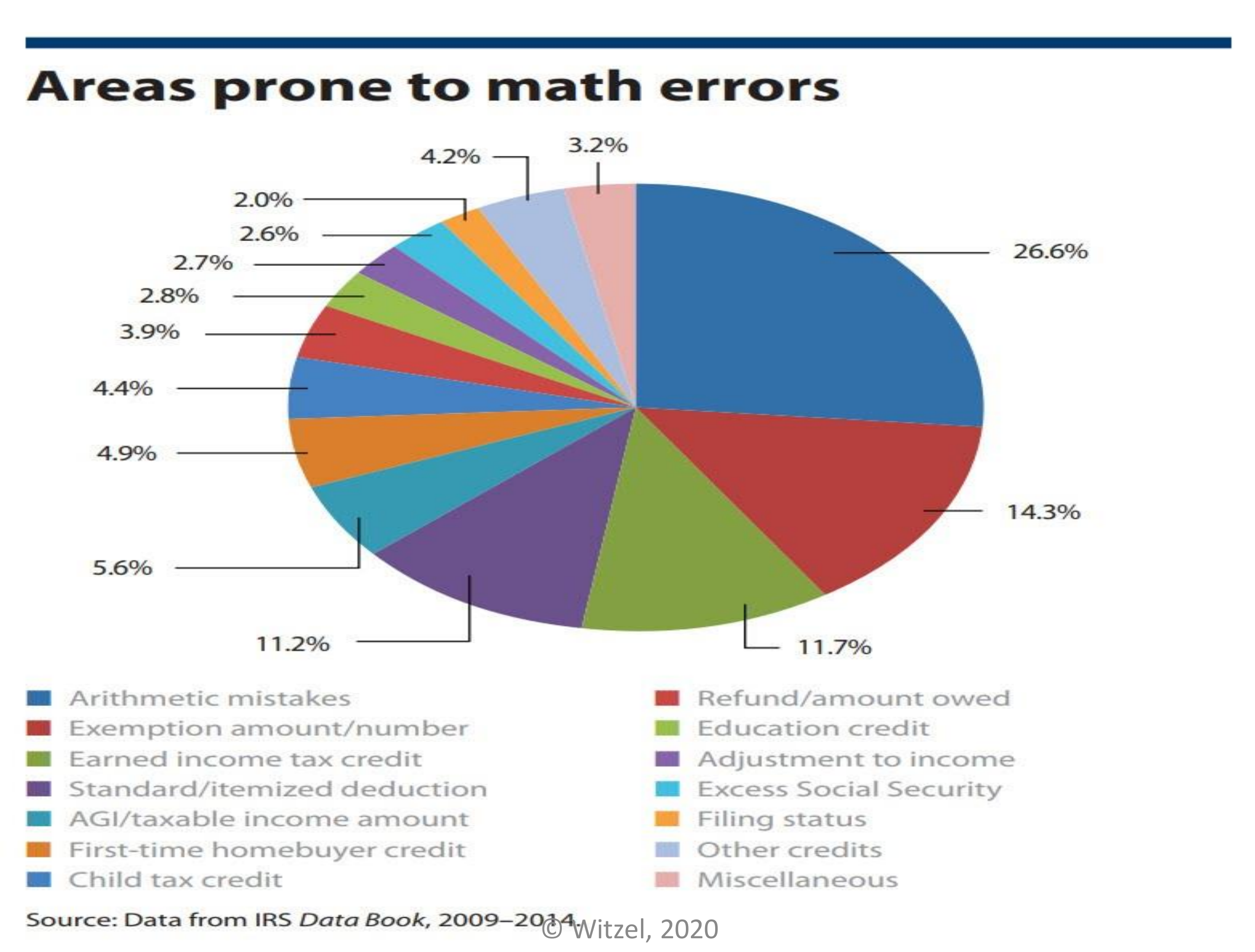
Which of the following best expresses your belief?

A. Students can succeed in secondary and advanced mathematics without computational fluency.

B. Students will struggle with secondary and advanced mathematics without computational fluency.

C. Students cannot succeed in secondary and advanced mathematics without computational fluency.

Accountancy Errors with Taxes



Agenda

- Identifying the need for computational proficiency
- Defining computational proficiency
- Strategies for Whole and Rational Number fluency
- Generalization (Embedded fluency)

Many children are unsuccessful learning math

(Nation's Report Card)

Grade and Content	National Average AT OR ABOVE PROFICIENT	Students with Disabilit[illegible]es AT OR [illegible] PROF[illegible]
Mathematics 4th Grade	40%	16[illegible]
Mathematics 8th Grade	34%	8%
Reading 4th Grade	36%	12%
Reading 8th Grade	35%	8%

Grade and Content	National Average BELOW BASIC	Students with Disabilities BELOW BASIC
[illegible]	[illegible]%	45%
Math[illegible] 8th Grade	[illegible]7%	68%
Reading 4th Grade	31%	67%
Reading 8th Grade	24%	63%

We need improved instruction and interventions

Mathematics: Learning Trajectory

- Preschool numerical understanding (e.g, estimation) and early grade growth in mathematics predicts math achievement through age 15 (Watts et al, 2014).
- Kindergarten early numeracy understandings and gains predicts mathematics and literacy knowledge five years later (Geary, 2011).
- "Knowledge of fractions and division predicts students' knowledge of algebra and high school math up to 6 years later" (Siegler et al., 2012)
- Eighth grade math may predict some of Algebra 2 success
    - "Less than 4% of students with high *grade 8 math* achievement failed Algebra II,
    - whereas 22% of students with low *grade 8 math* achievement failed it" (Stoker, Mellon, Sullivan, 2018)
- Algebra 2 completion predicts college and technical school completion (Moore & Shulock, 2010).

Why teach Whole number operations?

What math is needed to compute division of fractions?

Ex. $\frac{3}{8} \div \frac{2}{5} = \frac{15}{40} \div \frac{16}{40} = \frac{15/16}{1} = \frac{15}{16}$

Longitudinal data (Bailey, Siegler, & Geary, 2014):

- Whole number magnitude knowledge in first grade predicted knowledge of fraction magnitudes in middle school.
- Knowledge of whole number arithmetic in first grade predicted knowledge of fraction arithmetic in middle school

What do we do in the meantime for students who haven’t mastered their facts?

- Calculator?
  - “The Panel cautions that to the degree that calculators impede the development of automaticity, **fluency in computation will be adversely affected**” (NMAP, 2008, p. xxiv)
  - “**No different pattern of benefit from using calculators** was demonstrated for solving computation or word problem item types correctly” (Bouck, Bouck, & Hunley, 2015).

Overloading cognition in complex problems

"Students who can complete basic math computations problems with rapidity are likely to expend less time and effort on [more advanced] math activities and have less math anxiety"

(Parkhurst et al., 2010, p. 111).

"Without the ability to retrieve facts directly or automatically, students are likely to experience a high cognitive load as they perform a range of complex tasks" (Woodward, 2006, p. 269).

"Those with greater basic-fact fluency are more likely to choose to engage in math activities, which further enhance skills."
(Parkhurst et al, 2010)

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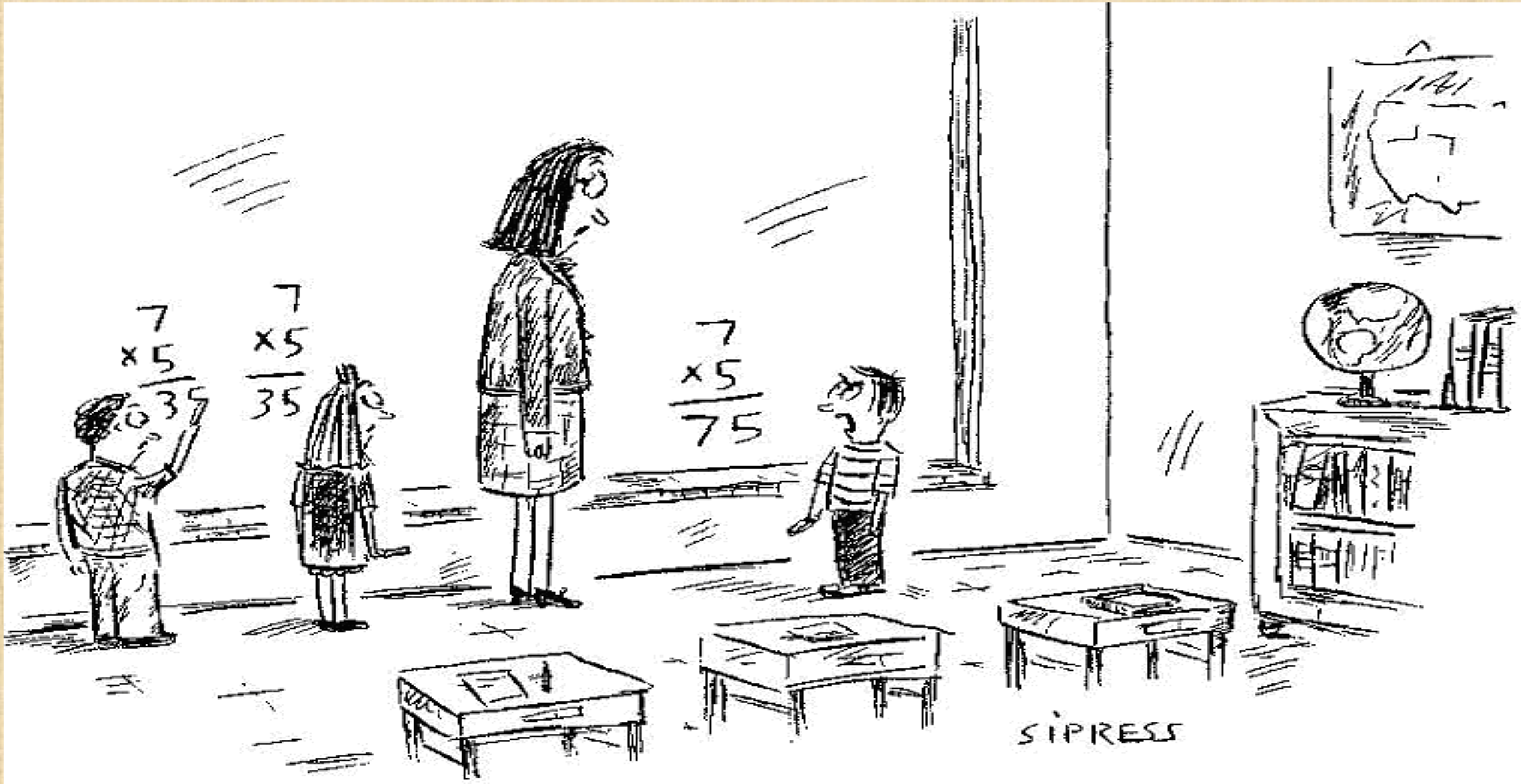
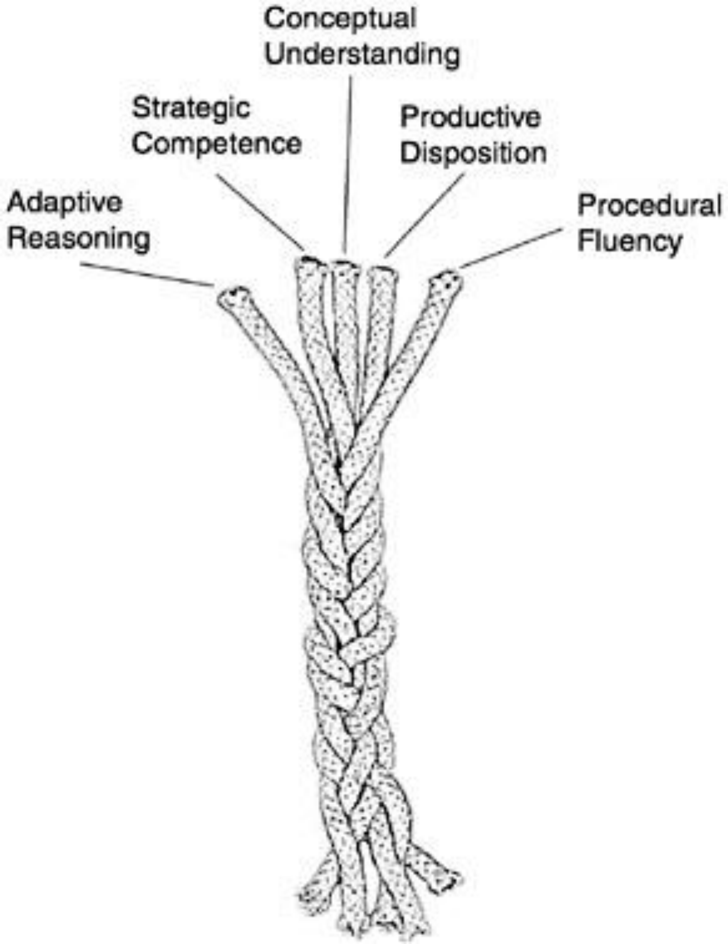


"It may be wrong, but it's how I feel."

Strands of Mathematical Proficiency (Adding It Up, 2001)



Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (KSDE, 2013)

Procedural fluency includes skills in carrying out procedures flexibly, fluently, and efficiently

Stages of computational proficiency development

A. Understanding key concepts

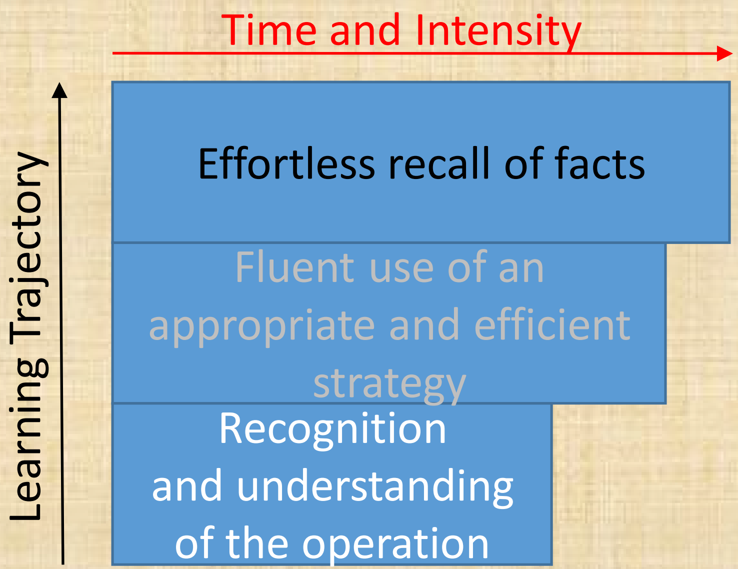
Teach students to recognize and explain operational symbols

B. Fluent use of strategies

Teach efficient and appropriate computational strategies

C. Recall / Automaticity of facts

Practice incremental recall of facts



Computational facility: A connected reality

- "There was a significant positive relationship between mathematics procedural understanding and conceptual understanding" (Ghazali et al., 2011, p. 684).
- "High amounts of procedural instruction were associated with higher calculation and conceptual math skills" (Bachman et al., 2015, p. 1-2).
- "Use should be made of what is clearly known from rigorous research about how children learn, especially by recognizing a) the advantages for children in having a strong start; b) <u>the mutually reinforcing benefits of conceptual understanding, procedural fluency, and automatic (i.e., quick and effortless) recall of facts</u>; and c) that effort, not just inherent talent, counts in mathematical achievement" (NMP, 2008, xiv).

Defining Fluency into Automaticity

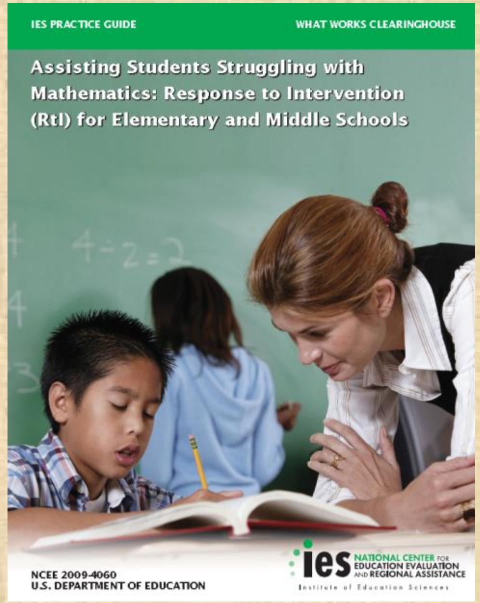
- Fact recall standards occur only after adequate groundwork in earlier grades on strategies and algorithms based on place value and the properties of operations.
- Automaticity is evident when a student solves a problem faster through recall than performing a mental algorithm (Burns, 2005).
- Speed is necessary but not sufficient for proficiency.

Practice to gain fluency and automaticity

- The RtI Panel (Gersten, Beckman, Clarke, Foegen, Marsh, Star, & Witzel, 2009) concluded that all students (K-8) receiving interventions should receive at least 10 minutes of practice per day in fact fluency.

  - K-5 should focus on whole numbers
  - 4-8 should focus on rational numbers
  - 9-12 should focus on rational numbers and embedded fluency

Consider both written and verbal output



Agenda

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- Strategies for Whole and Rational Number fluency
  - Understanding
  - Fluency
  - Automaticity (Fast Recall)
- Generalization (Embedded fluency)

Understanding Using Multiple Representations

- Three stages of learning

  C = Learning through concrete hands-on manipulative objects
  V = Learning through visual forms of the math skill
  A = Learning through work with abstract (Arabic) notation

- Aimed at teaching efficient and effective procedural fluency

- See IRIS Modules (https://iris.peabody.vanderbilt.edu/module/math/cresource/q2/p05/#content)

  or request videos from Dr. Witzel

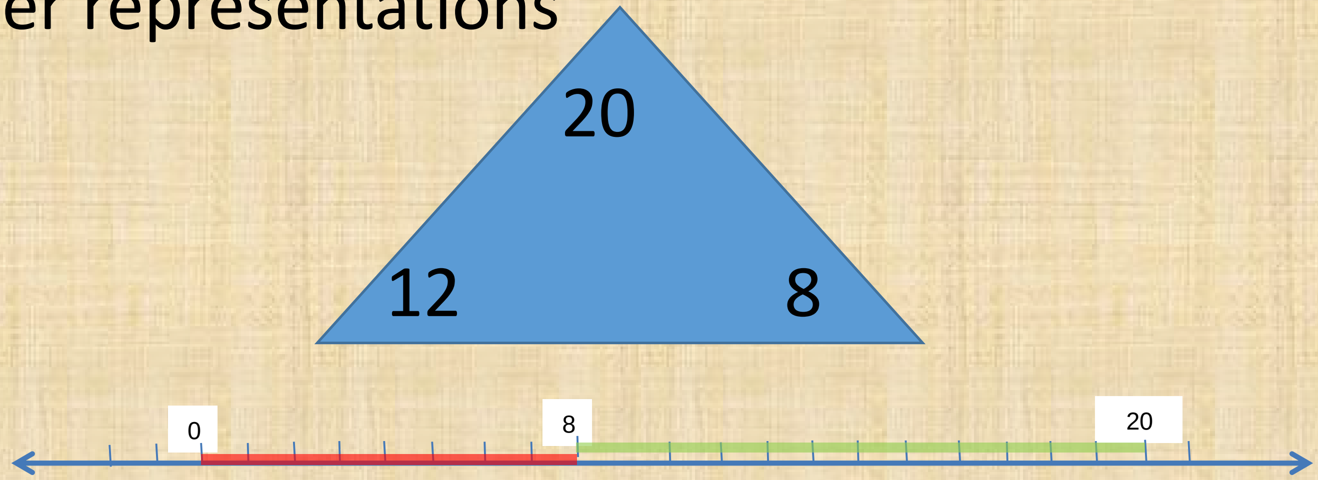
Number Lines

Whole numbers and fractions do not share properties of increase and decrease

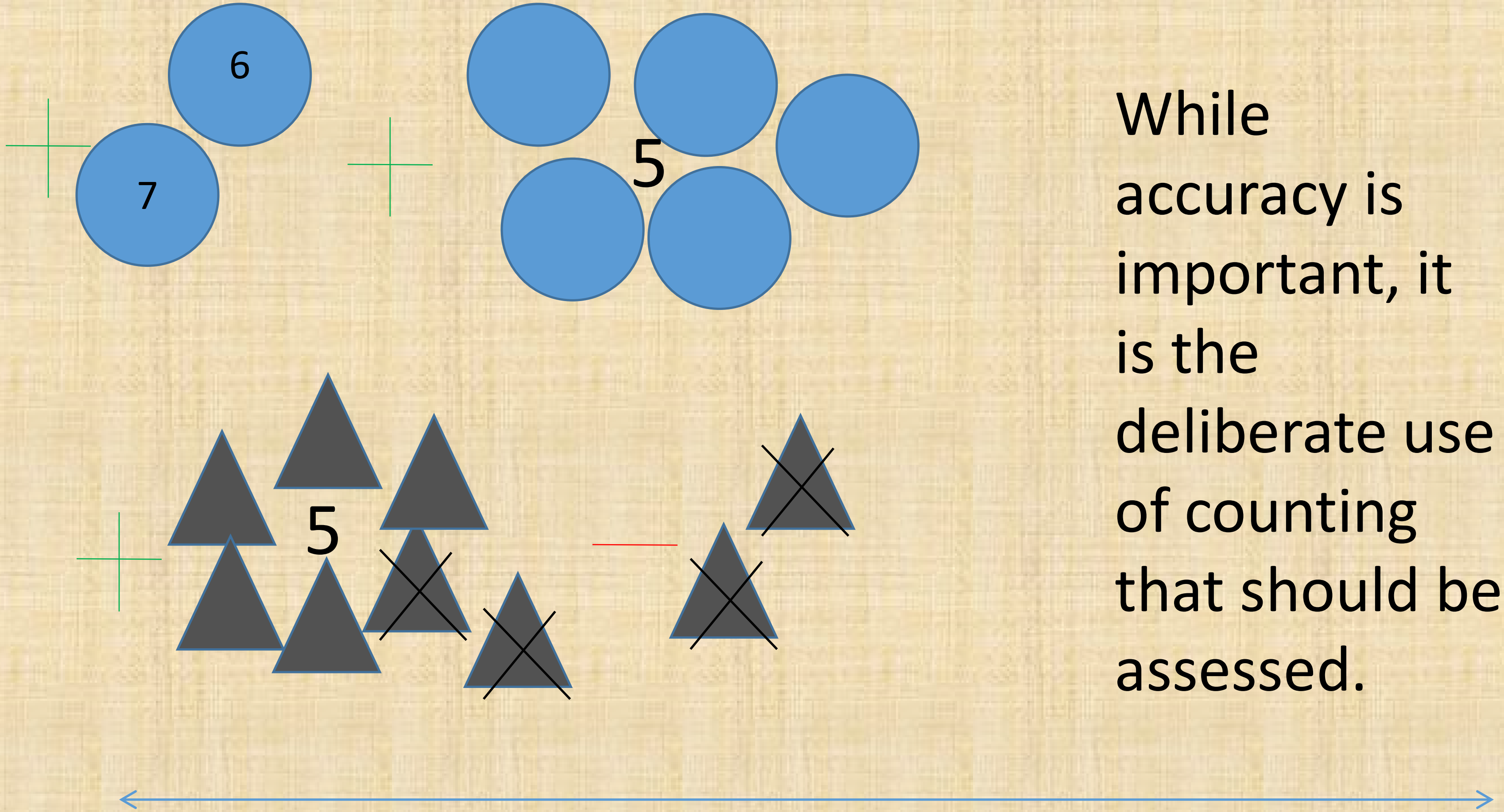
However, both whole numbers and fractions share a representation of magnitude

Use number lines physically and pictorially

Connect to other representations



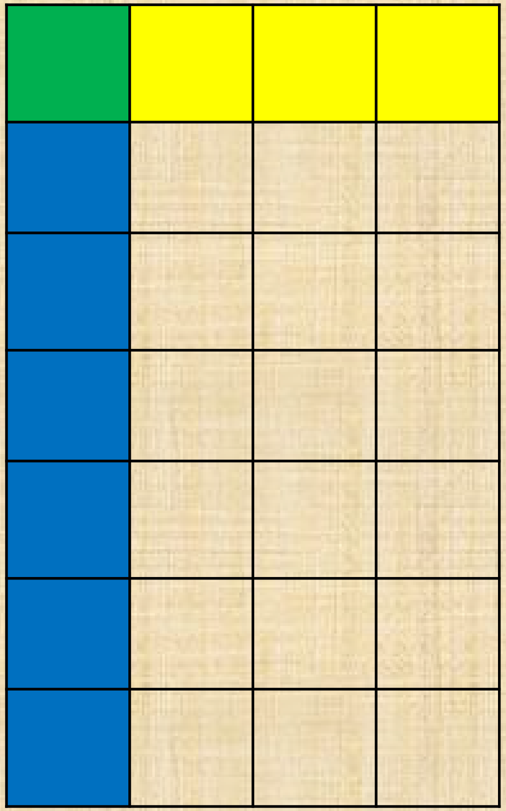
Addition and Subtraction

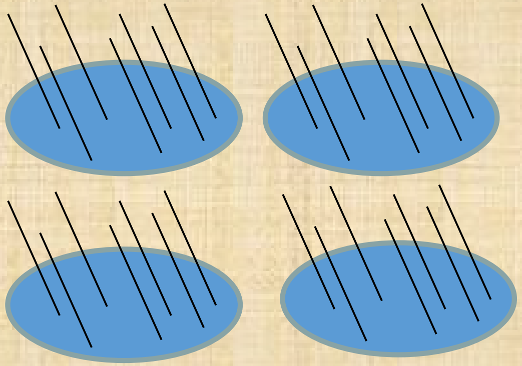


While accuracy is important, it is the deliberate use of counting that should be assessed.

Multiplication and Division

4 times tables





4(7) = 28; 4x7=28

$$\begin{array}{r} 4 \\ \underline{\times\ 7} \\ ? \end{array}$$

"Four groups of seven each totals how much?"

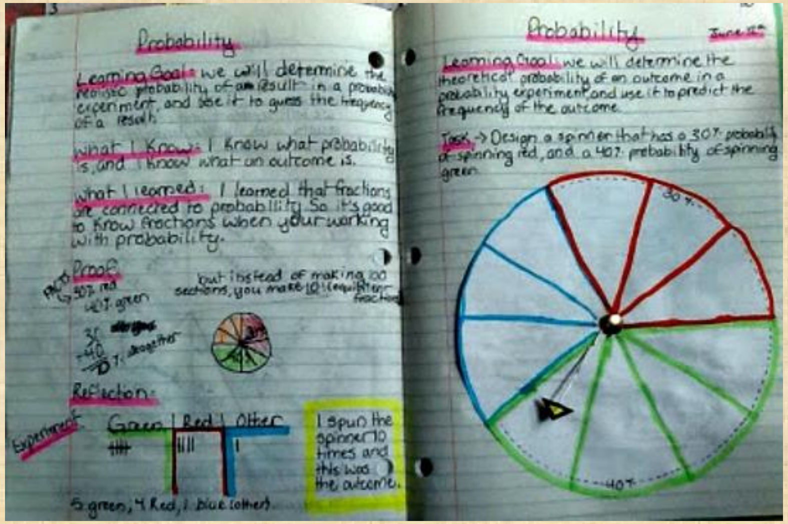
Manipulatives Multiplication

- Demo

Fluency requires an efficient strategy

Addition Fact Building Strategies

- 1 more
- 2 more
- Doubles
- Combinations of 10
- Commutativity
- By 10s
- By 5s

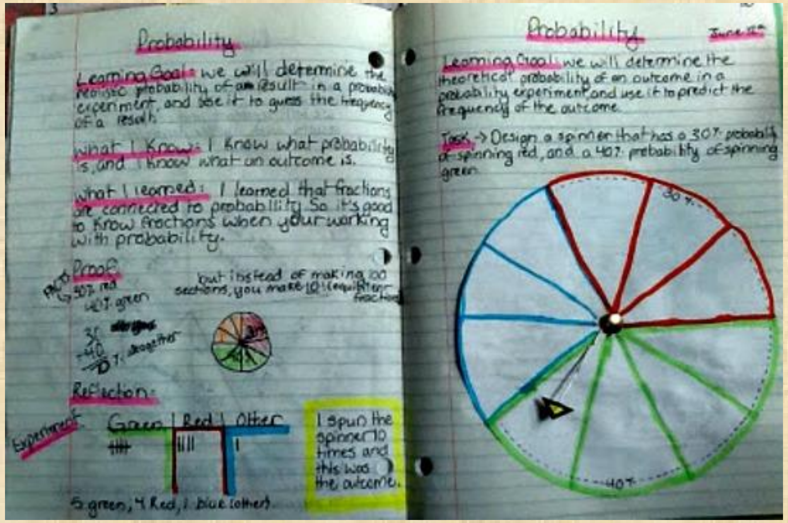


Adapted from Kling & Bay-Williams (2014)

Fluency requires an efficient strategy

Multiplication Fluency Strategies

- x0 and x1
- Double
- x10
- Multiples Counting
- Commutative property
- Compensation
- Two Hand Rule



Adapted from KSDE (2014)

How do we focus our efforts?

A. I am only responsible for my grade/course level standards. I don't have time for precursor skills.
B. I will help with some of the precursor skills but I highly prioritize my grade/course level standards.
C. I am responsible for preparing my students to future learning. Therefore, I focus on whatever math skills are missing in order to best learn my grade/course level standards.

Scope and Sequence of Multiplication

Kindergarten – Numbers and early strategi… unting (Number Sense)

1st grade – Counting by 10s, 2s, 5s, (…

2nd grade – Completing the m… …ction to "times" and groups; Missing factors

3rd grade – Single … …tion to automaticity; one-digit x two-digit strategies

4th grade – multi-dig… multiplication to 2x3 strategies

5th grade – multi-digit multiplication to 3x3 strategies

6th grade – Rational number strategies

"Learning your time tables isn't only for Third-Grade"

Fluency to Automaticity

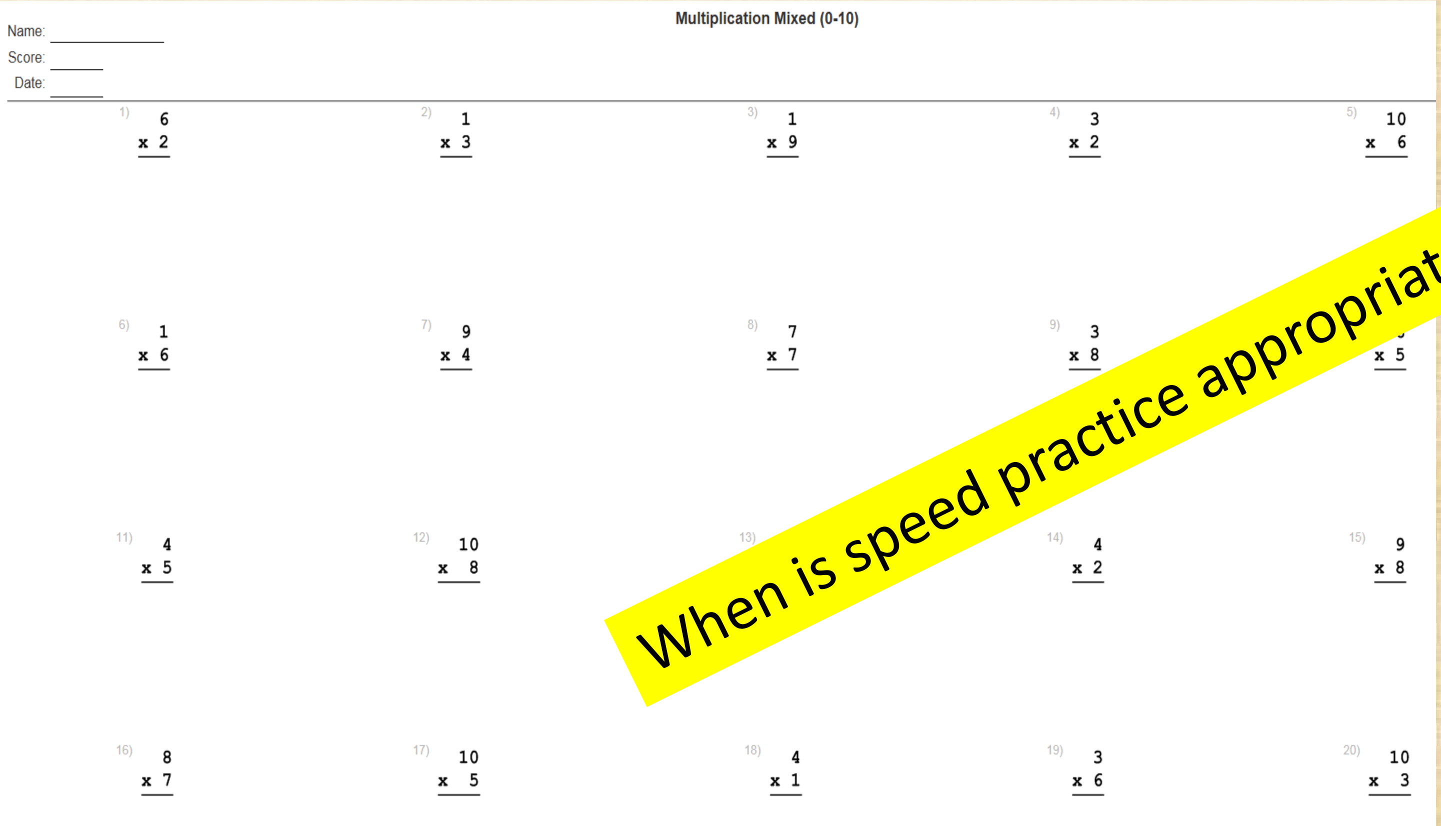
Fluency

- Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently (NRC, 2001)

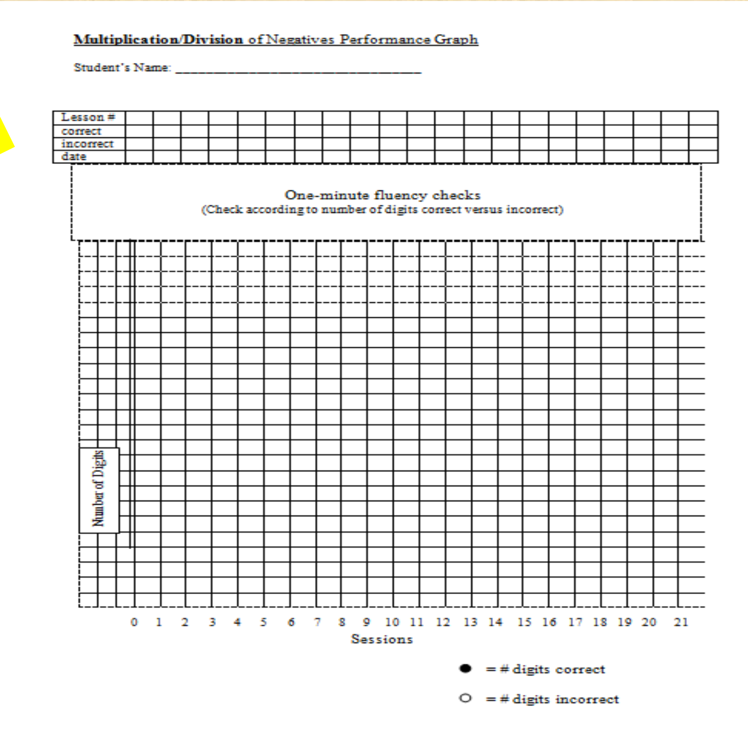
Automaticity

- Fast recall of a facts which seemingly appear instant
- Developmental Sequence: Conceptual understanding; Accuracy and reasoning; Fluency strategies; Automaticity (Riccomini, 2013)

Worksheet Practice: Popular but inefficient and inadequate by itself



When is speed practice appropriate?



https://www.superkids.com/aweb/tools/math/

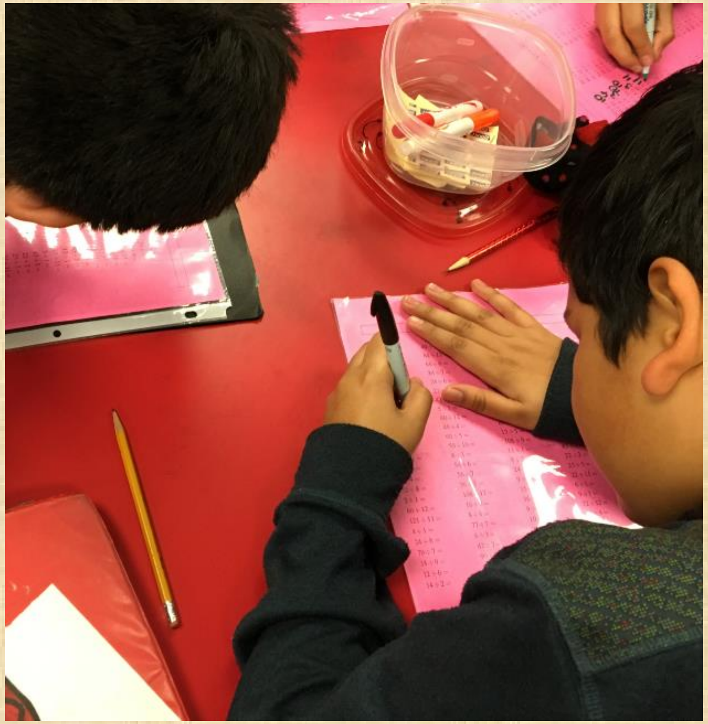
www.interventioncentral.org

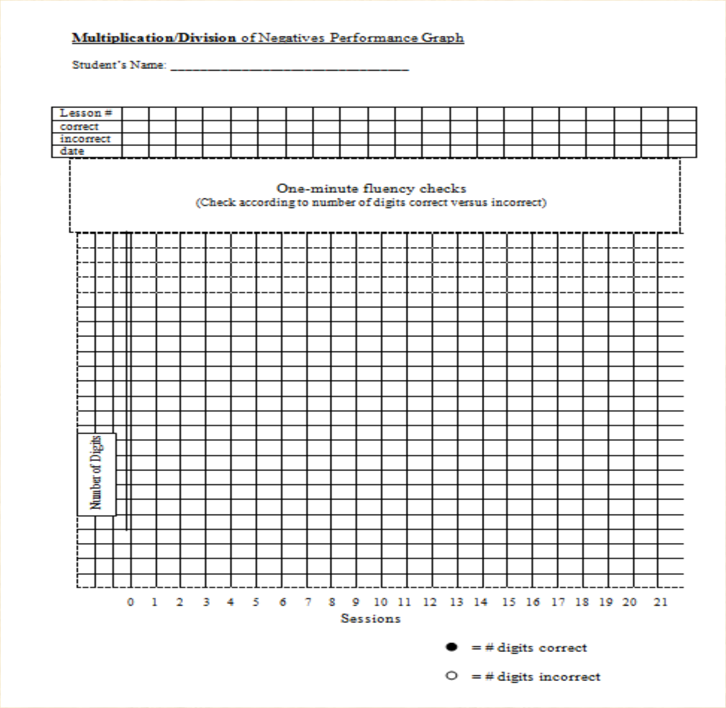
Witzel Negatives Multiplication / Division 07

Mixed signs with products less than 5

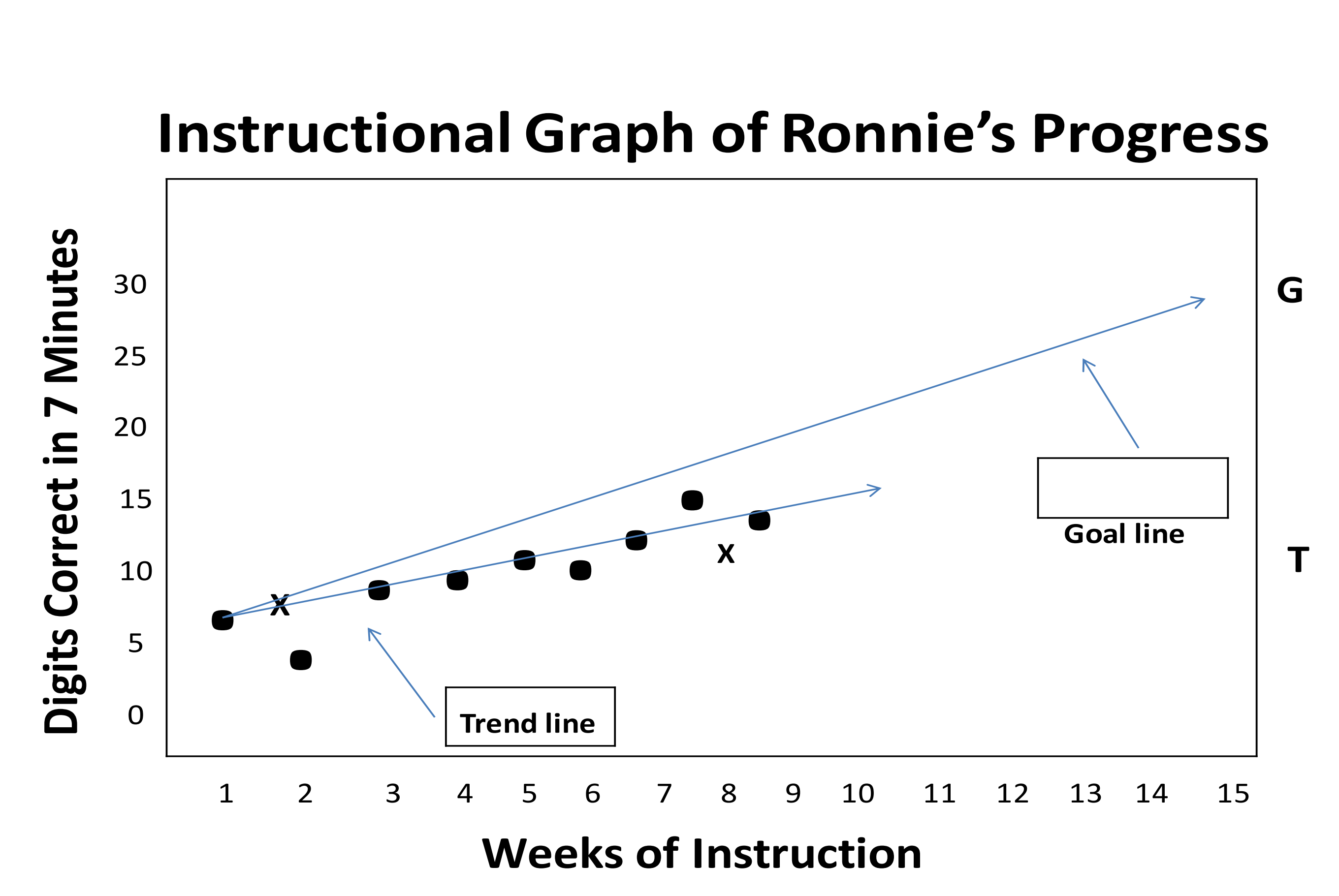
$\frac{(^{+}8)}{(^{+}2)}$	$\frac{(^{-}12)}{(^{-}3)}$	$(^{-}5)(^{+}5)$	$(^{-}4)(^{+}4)$
$(^{-}5)(^{+}6)$	$(^{-}2)(^{+}3)$	$\frac{(^{+}3)}{(^{+}3)}$	$(^{-}5)(^{-}2)$
$\frac{(^{+}8)}{(^{-}4)}$	$(^{+}3)(^{-}5)$	$(^{+}4)(^{+}3)$	$\frac{(^{-}10)}{(^{-}5)}$
$(^{-}1)(^{+}4)$	$\frac{(^{-}16)}{(^{+}4)}$	$(^{-}5)(^{+}4)$	$\frac{(^{+}12)}{(^{-}4)}$
$(^{+}5)(^{+}3)$	$\frac{(^{+}6)}{(^{-}3)}$	$\frac{(^{+}15)}{(^{+}5)}$	$(^{-}4)(^{+}2)$
$(^{-}2)(^{-}4)$	$\frac{(^{+}8)}{(^{-}2)}$	$(^{-}1)(^{+}5)$	$\frac{(^{-}9)}{(^{-}3)}$
$\frac{(^{-}5)}{(^{-}1)}$	$(^{+}3)(^{-}5)$	$\frac{(^{-}8)}{(^{+}4)}$	$\frac{(^{+}4)}{(^{+}1)}$
$(^{+}2)(^{+}5)$	$\frac{(^{-}15)}{(^{-}3)}$	$\frac{(^{-}5)}{(^{+}5)}$	$(^{+}5)(^{-}2)$
$(^{-}3)(^{+}4)$	$\frac{(^{+}4)}{(^{-}2)}$	$\frac{(^{+}16)}{(^{+}4)}$	$(^{-}3)(^{-}1)$
$(^{-}4)(^{-}2)$	$(5)(^{+}3)$	$\frac{(^{-}9)}{(^{+}3)}$	$(^{+}4)(^{+}4)$
$(^{+}2)(^{+}4)$	$(^{+}3)(^{+}3)$	$\frac{(^{-}6)}{(^{-}2)}$	$(^{+}4)(^{-}2)$

If you must use worksheet:
Go - Grade – Graph (Louderback, 2017)





Progress Monitoring (Witzel & Little, 2016)



Incremental Rehearsal

https://www.youtube.com/watch?v=0uVlF3hN1Uw

1. Traditional worksheet practice relies almost solely on memory skills.

   (34% of the variance in the scores)

2. IR is more systematic and relies much less on memory

   (2% of the variance in the scores)

Burns et al (2019)

Incremental Rehearsal: Visually and Auditory

Ratio

- Flash cards can be effective at establishing fluency and automaticity when using incremental rehearsal (Burns, 2005; Burns et al., 2014)
- Students are presented known to unknown material in a ratio of 9:1 **(90% to 10%)**
- In a ten card stack, this means that 9 of the answers are known and only one is yet to be learned.

Presenting unknown problems

- Students must build momentum and motivation by answering several correct in a row before an unknown problem is presented.
- When the unknown problem is presented, the answer is immediately provided.

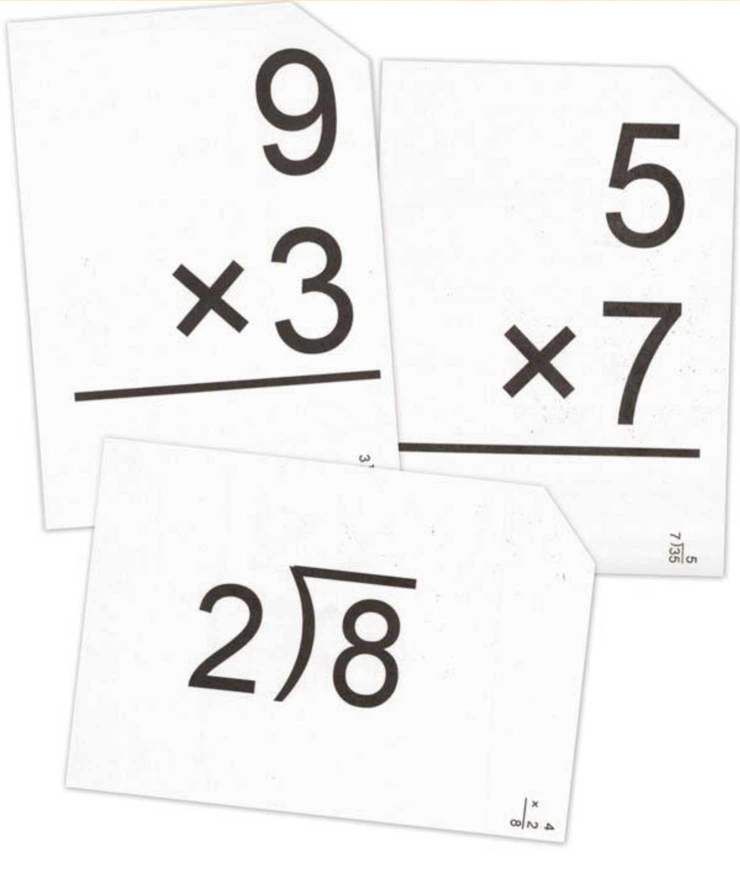
Incremental Rehearsal

Organizational structure of facts

Fluency charts can be developed from this approach

1. First unknown fact, first known fact
2. First unknown fact, first known fact, second known fact
3. First unknown fact, first known fact, second known fact, third known fact
4. First unknown fact, first known fact, second known fact, third known fact, fourth known fact
5. First unknown fact, first known fact, second known fact, third known fact, fourth known fact, fifth known fact
6. First unknown fact, first known fact, second known fact, third known fact, fourth known fact, fifth known fact, sixth known fact
7. First unknown fact, first known fact, second known fact, third known fact, fourth known fact, fifth known fact, sixth known fact, seventh known fact
8. First unknown fact, first known fact, second known fact, third known fact, fourth known fact, fifth known fact, sixth known fact, seventh known fact, eighth known fact

(Burns et al, 2019)



Hundreds Table Accommodation

- Mix the accommodation with fluency intervention
- Slowly fade the utility of the table by covering what has been "mastered"
- Make the table more cumbersome to use as the student progresses

Hundreds Table
A ~~Modification~~ Accommodation

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Hundreds Table Accommodation: Step 1

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Hundreds Table Accommodation: Step 2

The student learned 1x and 10x

	1	2	3	4	5	6	7	8	9	10
1										
2		4	6	8	10	12	14	16	18	
3		6	9	12	15	18	21	24	27	
4		8	12	16	20	24	28	32	36	
5		10	15	20	25	30	35	40	45	
6		12	18	24	30	36	42	48	54	
7		14	21	28	35	42	49	56	63	
8		16	24	32	40	48	56	64	72	
9		18	27	36	45	54	63	72	81	
10										

Hundreds Table Accommodation: Step 3

The student learned 1x, 10x, and 5x

	1	2	3	4	5	6	7	8	9	10
1										
2		4	6	8		12	14	16	18	
3		6	9	12		18	21	24	27	
4		8	12	16		24	28	32	36	
5										
6		12	18	24		36	42	48	54	
7		14	21	28		42	49	56	63	
8		16	24	32		48	56	64	72	
9		18	27	36		54	63	72	81	
10										

Hundreds Table Accommodation: Step 3+

The student learned 1x, 10x, 5x, and others

	1	2	3	4	5	6	7	8	9	10
1										
2						12	14		18	
3				12		18	21	24	27	
4			12	16		24	28	32	36	
5										
6		12	18	24		36	42	48	54	
7		14	21	28		42	49	56	63	
8			24	32		48	56		72	
9		18	27	36		54	63	72		
10										

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- Generalization (Embedded fluency)

What is Generalization?

- Students' ability to perform targeted skill under untrained and novel conditions
- Three dimensions:
  - Stimulus: Vertical addition → Horizontal addition
  - Response: Computer-Based Response →Pencil-and-Paper
  - Time: Fall 2019 → Fall 2020

(Cooper et al., 2007)

- Fluency (Increased Practice/Repetition) → Generalization

Two types of fluency: Digit computation and embedded fluency

- Digit computation: typically practiced in the form of basic problems where students do not need to analyze anything but the computational symbol (+, -, x, ÷)
- Embedded fluency: the ease with which computation can be performed within a complex math sentence.
  - Ex. In the math sentence 6x=42, a student must divide each side by 6 to determine that 1x=7.
  - In order to help students improve their embedded fluency, it is important to practice computation within forms of mathematics sentences, such as fractions, integers, distributive properties, and problems requiring understanding of the order of operations.

Probe 2. Solving inverse operations (fractional coefficients, add, subtract)
Lessons 4-7

a) $5 = Y - 8$	b) $3X = 27$	c) $4 + N = 5$
d) $\frac{N}{8} = 6$	e) $16 = Y + 7$	f) $25 = 5Y$
g) $X - 11 = 2$	h) $9 = 3Y$	i) $\frac{X}{2} = 6$
j) $P - 6 = 13$	k) $-3 = \frac{N}{9}$	l) $14 = 2 + T$
m) $13 = X - 14$	n) $64 = 4X$	o) $\frac{P}{6} = 7$
p) $P - 18 = 8$	q) $54 = 14 + M$	r) $\frac{K}{7} = 6$
s) $X + 8 = 26$	t) $63 = 9Y$	u) $19 = X - 3$

Embedded Fluency

How would you scaffold these problems for students with weaknesses in multiplication?

Simplify the expression,

$$\frac{2(8y^7 + 5x - 8)}{8x^2 + 6y^8}$$

Solve for *y*, 6*x*(3*y* – 6) – 5(7*y* + 1*y*) = -4(8 – *x*)

Daily routine for fact interventions

- Dedicate at least 10 minutes daily (Gersten et al, 2009)
- **Fluency is more than practice, it is instructional**
- Students may not know the reasoning behind computation

10 n – 3 – 1n, simplify the expression

2 minutes – strategy instruction

5 minutes –strategy practice

3 minutes – embedded instruction and practice

Repair as you go

- If grade level standard applies to your grade, make fluency strategies part of a routine until mastered
- If not on grade level, then
  - Investigate why errors develop
  - If fluency is an issue, then pinpoint how to teach the standard with computation that has been mastered

    3x rather than $\frac{2}{3}x$
  - Connect intervention focus to the standard, as much as possible

    Embedded fluency, -6(8x) for multiplication practice with middle level students
  - Individualize your preparations as much as possible (See Task Analysis PD)

What operations are included in these problems?

A. Solve 34 x 12

B. Solve: $8\overline{)312}$

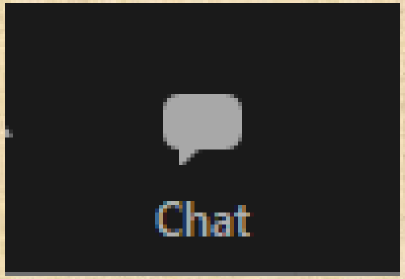
C. Simplify the expression: $2x - 6x + 7y + 3y$

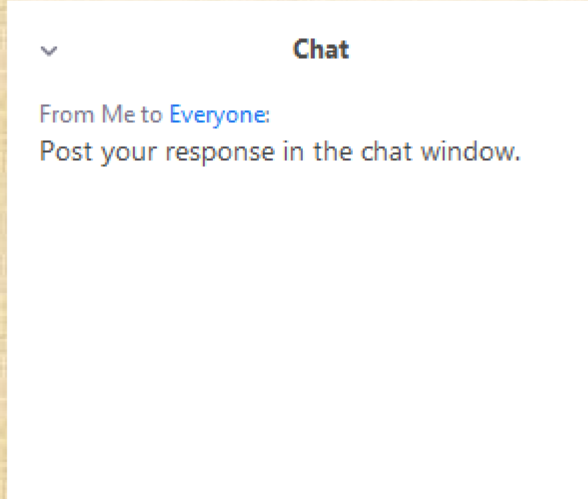
D. Simplify the expression: $6(7x\ - 8y)$

E. Solve this equation in slope-intercept form: $10x\ - 6 = 2y + 8$

Question / Poll

- Provide some links to excellent instructional ideas.





Sample Resources

Burns, M., Ysseldyke, J., Nelson, M., & Kanive, R. (2014). Number of repetitions required to retain single-digit multiplication math facts for elementary students. *School Psychology Quarterly*.

Parkhurst, J., Skinner, C. H., Yaw, J., Poncy, B., Adcock, W., & Luna, E. (2010). Efficient classwide remediation: using technology to identify idiosyncratic math facts for additional automaticity drills. *International Journal of Behavioral Consultation & Therapy, 6*(2), 111–123.

Witzel, B. S. (Ed.). (2016). *Bridging the gap between arithmetic and algebra.* Arlington, VA: Council for Exceptional Children.

Witzel, B. S., & Little, M. E. (2016). *Teaching elementary mathematics to struggling learners.* New York: Guilford.

Woodward, J. (2006). Developing automaticity in multiplication facts: Integrating strategy instruction with timed practice drills. *Learning Disabilities Quarterly, 29*, 269-289.