Rational Number Acquisition: A Focus on Fractions and Decimal-Fractions

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Rationale for the Rational

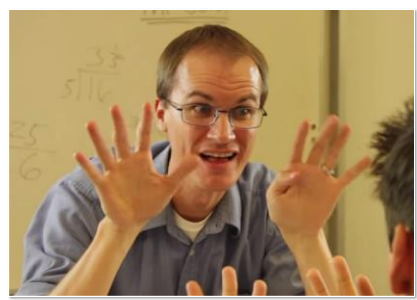
Low achievement in mathematics is a significant concern, especially in the area of rational numbers. Knowledge of fractions and decimals impacts a student's future mathematics performance. In this session, Dr. Witzel will provide 5 recommendations for effectively teaching students concepts related to rational numbers, including those who require additional assistance.

Rational Numbers are Common Difficulties

(Sanders, Riccomini, & Witzel, 2005)

Code	Category	Entering Math Tech 1	Entering Algebra 1
FRAC	Fractions and their Applications	3 (3.6%)	43 (44.8%)
DECM	Decimal-Fractions, their Operations and Applications: Percent	11 (13.1%)	64 (66.7%)
EXPS	Exponents and Square Roots; Scientific Notation	27 (21.1%)	62 (64.6%)
GRPH	Graphical Representation	13 (15.5%)	59 (61.5%)
INTG	Integers, their Operations & Applications	27 (32.1%)	83 (86.5%)
	Total Number of Students per course	**84**	**96**

It All Starts with Instruction!



Name the Most Common Answer

(Ryan & McCrae, 2005)

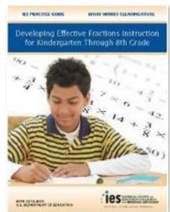
I) 0.3 x 0.24
- a) 0.072
- b) 0.08
- c) 0.72
- d) 0.8
- e) 7.2

I) 0.3 x 0.24

Response	Inferred Misconception	Frequency
a) 0.072	CORRECT	36.1%
b) 0.08	0.3 is one-third or the decimal implies division	3.5%
c) 0.72	3x24 and adjust to 2 decimal places	41.1%
d) 0.8	0.3 is one-third or a decimal implies division and adjust to 1 decimal place	2.8%
e) 7.2	0.3 x 0.24 = 3 x 2.4	15.3%
OMITTED		1.4%

Decimal-Fractions Practice Guide

1) Build on students' informal understanding of sharing and proportionality.
2) Teach that fractions are numbers. Use number lines as a central representational tool.
3) Teach why procedures for computations with fractions make sense
4) Develop conceptual understanding of strategies for solving ratio, rate, and proportion problems before or rather than short cuts.
5) Professional development programs should prioritize fractions understanding.



Name the Most Common Answer

(Ryan & McCrae, 2005)

II) 912 + $^{4}/_{100}$ in decimal form

a) 912.4
b) 912.04
c) 912.004
d) 912.25
e) 912.025

II) 912 + $^{4}/_{100}$ in Decimal Form

Response	Inferred Misconception	Frequency
a) 912.4	Hundredths is first decimal place	3.5%
b) 912.04	CORRECT	76.3%
c) 912.004	***Onesths***; Tenths, Hundredths	12.2%
d) 912.25	4/100 is ¼ or 100÷4 =1/25=0.25	6.0%
e) 912.025	100÷4=25 and ***onesths***, tenths, and hundredths	1.6%
OMITTED		0.7%

Name the Most Common Answer

(Ryan & McCrae, 2005)

III) 300.62 ÷ 100

a) 30062
b) 30.062
c) 30.62
d) 3.0062
e) 3.62

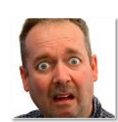
III) 300.62 ÷ 100

Response	Inferred Misconception	Frequency
a) 30062	Move the decimal point 2 places to the right	0%
b) 30.062	Move the decimal point 1 place to the left	6.4%
c) 30.62	Cancel the zero	2.6%
d) 3.0062	CORRECT	68.8%
e) 3.62	Integer-decimal separation or cancel 2 zeros	22.0%
OMITTED		0%

Warning!

Those errors were not made by P-12 students.

They were made by teacher candidates.



Common Difficulties with Fractions

(Riccomini, Hughes, Morano, Hwang, & Witzel, in-press)

Fraction Item Error Analysis of Low, Middle, and High Achieving Groups

	Achievement Groups										
	Low				Middle				High		
Item Categories	M	SD	Rank		M	SD	Rank		M	SD	Rank
Division	.03	.15	1		.16	.31	1		.64	.44	2
Ordering	.04	.23	2		.25	.44	2		.55	.50	1
Multiplication	.51	.39	8		.42	.40	3		.88	.23	5
Word Problems	.09	.15	5		.5	.33	4		.87	.20	4
Addition D(1)	.04	.17	3		.51	.42	5		.82	.30	3
Subtraction D(1)	.08	.26	4		.62	.43	6		.91	.24	6
Transform L(2)	.30	.33	6		.8	.29	7		.94	.15	7
Transform E(3)	.43	.47	7		.91	.25	8		.99	.10	9
Subtraction S(4)	.67	.47	9		.95	.22	9		.98	.13	8
Addition S(4)	.73	.37	10		.97	.15	10		.99	.08	10

Note. (1) Different denominator, (2) Least form, (3) Equivalent form, (4) Same denominator. Rank numbers from 1-10 signify greatest frequency of errors '1' to smallest frequency of errors '10'.

Pesky Remainders!

"John is taking 10 friends on a trip. Each car holds 3 people. How many cars will John need for his trip? Justify your answer."

- Explain the significance of the number of cars.
- Why not a fractional answer? Make sense of the mathematics

http://www.cpalms.org/Public/PreviewResource/Preview/29139

Picture This! Fractions as Division

"Maria was very excited to have earned an entire package of Starburst candies, so were all her friends. She wanted to show what a good friend she was and decided to share the package equally between her 6 friends and herself. She counted a total of 49 candies in the package. How many pieces of candy should each receive? How would you model this situation?"

http://www.cpalms.org/Public/PreviewResource/Preview/46576

Avoid Tricks: Converting Fractions

Convert this mixed fraction into an improper fraction.

$4\,{}^{2}/_{5}$

How did you know how to do it?

Did you…

a. 4x5

b. + 2 = 22

c. The 5 slides over

d. $^{22}/_{5}$

Why?

Say, "Four and two – fifths"

$^{4}/_{1} + {}^{2}/_{5}$ or $^{20}/_{5} + {}^{2}/_{5} = {}^{22}/_{5}$

Avoid Tricks: Division of Fractions

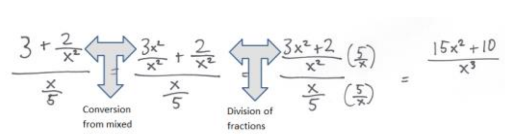
- Why is it that when you divide fractions, the answer might be larger? Moreover, why do you invert and multiply?

"Just flip it!"

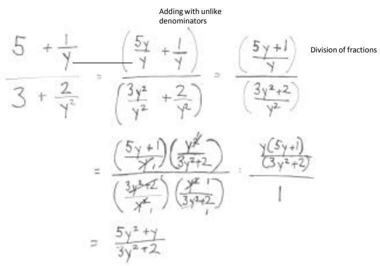
$^{2}/_{3} \div {}^{1}/_{4} = {}^{2}/_{3}\,({}^{4}/_{1}) = {}^{8}/_{3}$

$$\frac{{}^{2}/_{3}\,({}^{4}/_{1})}{{}^{1}/_{4}\,({}^{4}/_{1})} = \frac{{}^{8}/_{3}}{{}^{4}/_{4}} = \frac{{}^{8}/_{3}}{{}^{1}/_{1}} = {}^{8}/_{3}$$

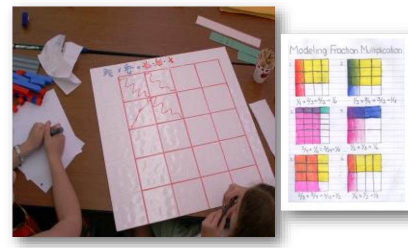
Why Understand Fractions?



Why Teach the Basics Correctly



(2/5) (2/4)



What Should We Do to Help?

- Learn how students tackle rational number problems.
- Help teachers learn gap skills within rational number naming, magnitude, and computation.
- Teach instructional and intervention strategies that help students learn decimals and fractions.

Decimal Subtraction (Boerst, 2015)

5 – 0.12

➢ Do the subtraction problem yourself.

Think-Pair-Share

What strategies could fifth graders use to solve this problem?

Using Representations to Support the Teaching and Learning of Decimals

A. Understanding decimals
   a) Challenges using generalizations from work with whole numbers
   b) Common student misconceptions
   c) Making connections with fractions
B. Representing decimals
   a) Choosing and using representations
C. Comparing decimals
   a) Modeling the comparison
   b) Choosing numerical examples
D. Representing and making sense of computation
   a) Analyzing common student errors
   b) Modeling computation with decimals
E. Intervention Bonus

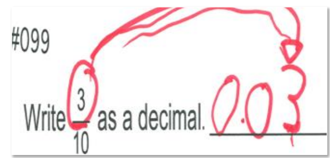
A. Understanding Fractions and Decimal-Fractions

What is Mathematically Challenging About Decimals?

Example of error	Underlying misconception
.34 > .5	Longer decimals are greater *(overgeneralizing from whole numbers)*
.75 > .7500002	Longer decimals are lesser *(overgeneralizing new insights into decimals)*
.20 is ten times greater than .2	Adding a zero to the right makes a number ten times larger *(overgeneralizing from whole numbers)*
Hundreds, tens, ones, oneths, tenths, hundredths…	Lack of understanding of the "specialness" of one in the place value system

Adapted from Boerst & Shaughnessy, 2015; Irwin, 2001

Common Misunderstandings About Connecting Fractions and Decimals

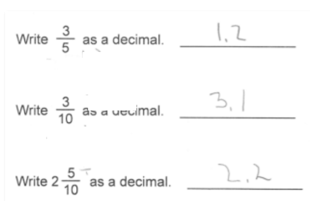


How might a student produce this answer?

Writing the numerator as the decimal

Common Misunderstandings About Connecting Fractions and Decimals

How might a students produce these answers?



Dividing the denominator by the numerator (and not knowing how to express the remainder)

Common Misunderstandings About Connecting Fractions and Decimals

- Separating the numerator and denominator with a decimal point
- Writing the numerator as the decimal
- Dividing the denominator by the numerator
- Ignoring whole numbers

How can we help student understand decimals as numbers?

Place Value Skills

27 = 2 tens and 7 ones

45 = 4 tens and 5 ones

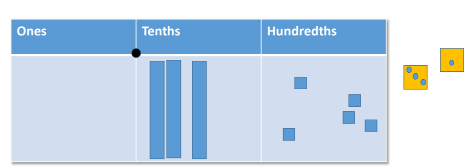
Hundreds	Tens	Ones

- Should be represented physically and verbally
- Advanced learners should use place value within a calculation exercise.

Ones	Tenths	Hundredths

Race to One

- Partner work - Repeat each new number as a total amount and place value increments



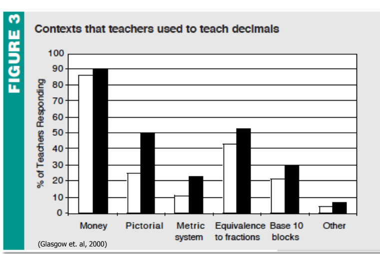
B. Representing Fractions and Decimal-Fractions

"[Fraction magnitude] knowledge often emerges through instruction and practice that helps children to map numerically expressed fractions (N_1/N_2) onto number lines, rectangles, and (especially) circles, for example, the omnipresent circular pizza representation" (Bailey et al., 2015, p. 80)

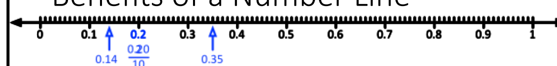
Commonplace/Typical Decimal Representations Used in Teaching

- What representations do teachers use to support students' understanding of fractions and fraction-decimals?
- What contexts do teachers use to support students' understanding of decimals? Why?

How do teachers represent fraction-decimals to students?



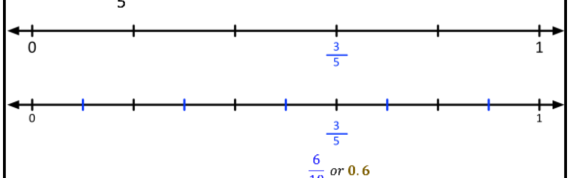
Benefits of a Number Line



- Represents decimals as numbers
- Addresses these (mis)-understandings and others:
  - ✓Longer decimals are greater
  - ✓Longer decimals are lesser
  - ✓Adding a zero to the right makes a number ten times larger
- Density of the rational numbers
- Infinitely many names for any point on the line

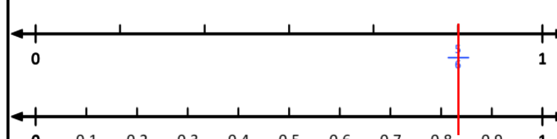
Connect Decimals and Fractions Using the Number Line

Write $\frac{3}{5}$ as a decimal.



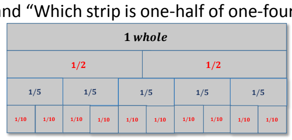
Connect Decimals and Fractions Using the Number Line

Identify $\frac{5}{6}$ as a decimal.



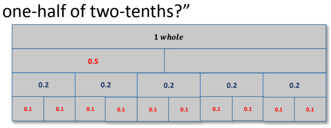
Whole to Part Thinking: Making and Investigating Fraction Strips

- Students cut, fold, and color strips of paper to create length-based models of fraction lines.
- Strips are stacked in order to make comparisons
- Ask questions such as, "Which strip is one-fourth of the whole?" and "Which strip is one-half of one-fourth?"



Whole to Part Thinking: Making and Investigating Decimal Strips

- Students cut, fold, and color strips of paper to create length-based models of decimals.
- Strips are stacked in order to make comparisons
- Ask questions such as, "Which strips show four-tenths of the whole?" and "Which strip is one-half?"
- "Which is one-half of two-tenths?"

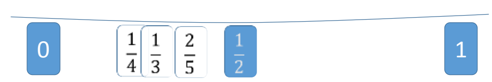


Whole to Part Thinking: Fraction and Decimal Strips

- Use the term "whole" rather than "one" so that students understand the proportionality of fractions per a whole.
- In pairs, have students communicate relationships.
- List all relationships on chart paper and have students confirm or deny these relationships.

http://www.cpalms.org/Public/PreviewResource/Preview/30161

Fractional Clothesline



Variations include:

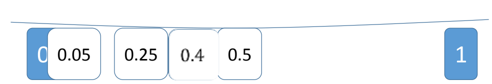
- Clothesline versus tape and sticky-notes
- Using key fraction benchmarks to assist students
- Graduating from whole class to small group or individual

Fractional Clothesline (cont.)

- Stretch a clothesline across the room.
- Pin cards to indicate location on a number line
- Vary cards between fractions, decimals, percents, and a combination
- Vary the objective from ordering to comparing
- Ask students to explain their reasoning

http://www.cpalms.org/Public/PreviewResourceUrl/Preview/5109

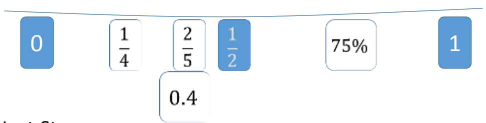
Decimal Clothesline: Extension



Next Steps:

a) fractions to decimals to percent
b) combinations of cards
c) change the representation of a whole

Decimal Clothesline: Extension



Next Steps:

a) fractions to decimals to percent

b) combinations of cards

c) change the representation of a whole

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The Challenges of Using the Number Line

- It is challenging for students to generate number line representations with parts smaller than tenths
- Pre-partitioned number lines can show parts smaller than tenths, but often require additional work to make the partitioning meaningful to students
- The amount of decimal places that you want to represent can constrain the span of numbers you are able to use

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From the IES Practice Guide

Recommendation #2:

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers.

Things to try :

- Use representations to support meaningful connection of fractions and decimals
- Analyze the affordances of different representations / contexts in light of mathematical purposes

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C. Comparing Decimals

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Ordering Decimals with Number Lines

Put the first string of decimals in order using a number line.

a) 2.3 0.23 0.8 0.08 .23

If you finish, try the second string:

b) 0.4 1.4 .55 .0098 11 0.40

- What did you notice about putting decimal numbers in order using the number line?
- What do the number lines show, and how do they help with the typical difficulties we discussed?

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Possible Observations

- The magnitudes of the decimals are visible
- The equivalence of multiple decimal representations is visible (it is the same point even when the partitioning looks different)
- Decimals need to be selected strategically
- A number line does not produce answers– students need to learn about its features and properties and develop ways of using them to do mathematical work

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Developing Tasks That Illuminate Big Ideas of Decimals

Develop a set of five numbers that would require that students grapple with decimal challenges and misconceptions through the use of number lines.

1.20 1.02 1.2 1.020 10.2	Longer decimals are greater (or longer decimals are lesser)
2.1 0.4 2.10 0.40 0.04	Adding a zero to the right makes a number ten times larger
0.123 0.4 .35 .456 0.5	You can ignore all zeros to the right of the decimal point

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From the IES Practice Guide

Recommendation #2:

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers.

Things to try :

- Generate numerical examples to support "productive struggle" with key decimal ideas and likely misconceptions
- Analyze the affordances of different representations / contexts for particular numerical examples

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D. Representing and Making Sense of Computation

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Meta-Analysis Findings for Algebraic Interventions (Hughes, Witzel, et al., 2014)

- Cognitive and model-based problem solving
  - ES= 0.693
- Concrete-Representation-Abstract
  - ES=0.431
- Peer Tutoring
  - ES=0.102
- Graphic Organizers alone
  - ES=0.106
- Technology
  - ES=0.890
- Single-sex instruction
  - ES=0.090

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CRA as Effective Instruction / Intervention

(Gersten et al, 2009; NMP, 2008; Riccomini & Witzel, 2010; Witzel, 2005)

Concrete to Representational to Abstract Sequence of Instruction (CRA)

- Concrete (expeditious use of manipulatives)
- Representations (pictorial)
- Abstract procedures

Excellent for teaching accuracy and understanding

Examples:

http://engage.ucf.edu/v/p/2wKBsbB

http://fcit.usf.edu/mathvids/strategies/cra.html

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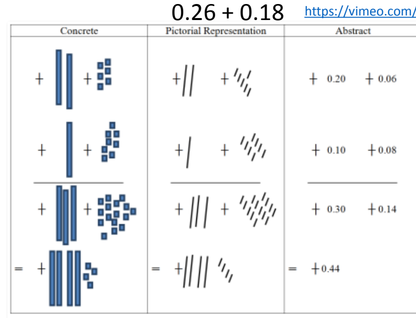
Use Place Value to Add Within 100

0.26 + 0.18 https://vimeo.com/128677958



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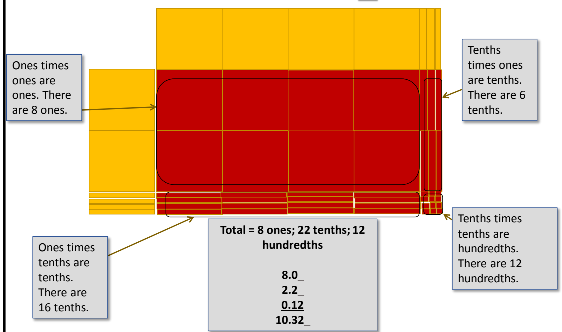
Adapted from (Witzel, et al. 2013)

(4.3)(2.4) Using CRA



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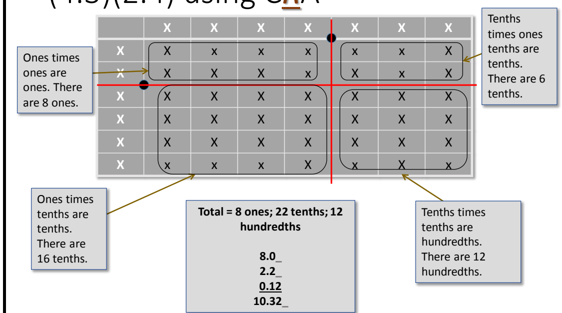
(4.3)(2.4) using CRA



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(4.3)(2.4) using CRA

multi	4	.3
2	8	0.6
.4	1.6	0.12

Total = 8 ones; 22 tenths; 12 hundredths

8.0_
2.2_
0.12
10.32_

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Revisiting the IES Practice Guide

Recommendation #3:

Help students understand why procedures for computations with fractions make sense.

Things to try:

- Use the representations to support sense making of computational procedures and solutions
- Don't race to commutativity, while answers will end up the same, the representations of the process will look different
- Develop/select problems that expose and work through common misconceptions

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Improving Instruction to Support Understanding of Decimals

To help students recognize that fractions (and decimals) are numbers and that they expand the number system beyond whole numbers, try to:

- Use representations to support meaningful connection of fractions and decimals
- Analyze the affordances of different representations / contexts in light of mathematical purposes
- Generate numerical examples to support "productive struggle" with key decimal ideas and likely misconceptions
- Analyze the affordances of different representations / contexts for particular numerical examples

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Improving Instruction to Support Understanding of Decimals

To help students understand why procedures for computations with fractions (and decimals) make sense.

- Use the representations to support sense making of computational procedures and solutions
- Don't race to commutativity, while answers will end up the same, the representations of the process will look different
- Develop/select problems that expose and work through common misconceptions

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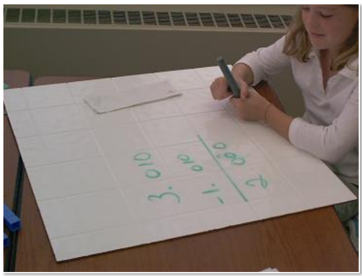
E. Fractions and Decimals - Interventions



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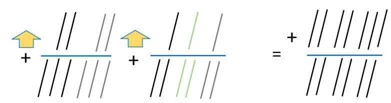
Intervention with Fractions Procedures

(Witzel & Riccomini, 2009)

$^{2}/_{3} + ^{1}/_{2}$



$\frac{(2+2)}{(3+3)} + \frac{(1+1+1)}{(2+2+2)} = \frac{7}{6}$

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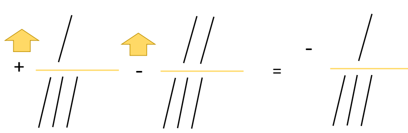
Aim Interventions at Procedural Processes

(Witzel & Riccomini, 2009)

$^{1}/_{3} - ^{2}/_{3}$



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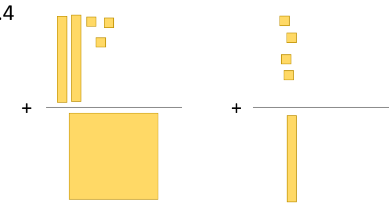
Connection to Decimals

(Witzel & Little, 2016)

- Fractions show the denominator while for decimals it is verbally interpreted.
- 0.23+0.4



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Connection to Decimals

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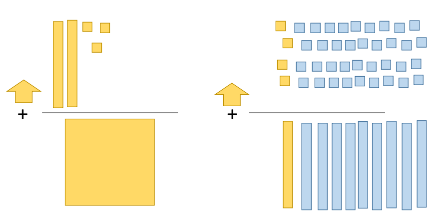

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Conclusion

- Balance understanding with representations of decimals
- Connect area and line models to make sense of fractions
- Use language and representations to aid computation practice of decimals

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