



Mathematical Fluency – Techniques, Access and Sustainability for All Students

Presented by

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www.Corelearn.com

CORE MATH[®]

Fluency – There's always your fingers...

Kid Snippets – "Math Class"

<https://www.youtube.com/watch?v=KdxEAt91D7k>



Essential Questions to Answer Today

- What is fluency and why is it important?
- What is a systematic pathway to fluency?
- What are some ideas, activities and resources to implement in the classroom to build fluency, number sense, and reasoning skills?





What Is Fluency?

. . . procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately) . . .

CCSSM, p. 6

BREAKOUTS:

Introduce yourself: name, location, role.

Share your thoughts on any of the following:

- What does fluency mean to you?
- What are the types of things in math that students should be fluent with?
- What is your favorite activity for building fluency and number sense?

Fluency Chart for K–8

Grade	Required Fluency
K	Add and subtract within 5
1	Add and subtract within 10
2	Add and subtract within 20 (mentally) Add and subtract within 100
3	Multiply and divide within 100 Add and subtract within 1,000
4	Add and subtract multidigit whole numbers using standard algorithms
5	Multiply multidigit whole numbers using standard algorithm
6	Add, subtract, multiply, and divide multidigit numbers (including decimals) using standard algorithms
6-8	Compute with positive and negative fractions and decimals



Retaining Math Knowledge: Memory

Limits of Working Memory

$$7(x + 8) = 7x + 56$$

Capacity and Time

Age (Years)	Average Capacity Range (Chunks of Information)	Average Time Limit for Retention
Younger than 5	2 ± 1	Unknown
From 5 to 14	5 ± 2	5 to 10 minutes
Older than 14	7 ± 2	10 to 20 minutes

David A. Sousa, *How the Brain Learns Mathematics*, 2008

Practice Is Necessary

Studies show that if material is studied for one semester or one year, it will be retained adequately for perhaps a year after the last practice (Semb, Ellis, & Araujo 1993), but most of it will be forgotten by the end of three or four years in the absence of further practice. If material is studied for three or four years, however, the learning may be retained for as long as 50 years after the last practice (Bahrick 1984; Bahrick & Hall 1991).

Daniel Willingham, *American Educator*, Spring 2004

Rote and Elaborate Rehearsal

- **Rote rehearsal:** Memorization without continuing to think through an idea or fact.
- **Elaborate rehearsal:** Making sense of ideas and information. The learner processes and reprocesses information to connect it together, to connect it to prior learning, and to assign meaning to it. Elaborate rehearsal is necessary for students to probe the deeper meaning and interrelationships of mathematical concepts.

Why We Need Elaborate Rehearsal

Memories are formed as the residue of thought. You remember what you think about, but not every fleeting thought—only those matters to which you really devote some attention.

Daniel Willingham 2008

Recommendations from Research

- *Provide 10 minutes of daily practice to strengthen needed fluency with facts and procedures.*

Gersten et al. 2009

- *Distributed or spaced practice, repeated practice of previously learned knowledge over “a long period of time,” has a high effect size of 0.71.*

Hattie, Fisher, and Frey 2017

- *Fluency with whole numbers and fractions are part of a critical foundation for learning algebra.*

National Mathematics Advisory Panel 2008

5 Fundamentals of Fact Fluency

Harvesting
Good
Thinking:
Planting
Seeds



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- #1: Mastery must focus on fluency!**
- #2: Fluency develops in three phases.**
- #3: Knowing foundational facts must precede derived facts.**
- #4: Timed tests do not assess fluency.**
- #5: Students need substantial and enjoyable practice.**

Bay-Williams, J., & Kling, G. (2019). *Math Fact Fluency: 60+ Games and Assessment Tools to Support Learning and Retention*. Alexandria, VA: ASCD.

Jennifer Bay-Williams Sept. 2021 CORE webinar *Harvesting Good Thinking: Growing Basic Fact Strategies to General Reasoning Strategies*

Fluency Develops in Three Phases

- **Phase 1: Counting**
(counts or skip counts)
- **Phase 2: Deriving**
(uses reasoning strategies based on known facts)
- **Phase 3: Mastery (Automaticity)**
(efficient production of answers)

Jennifer Bay-Williams Sept. 8, 2021 CORE webinar *Harvesting Good Thinking: Growing Basic Fact Strategies to General Reasoning Strategies*; (Based on Baroody 2006)

Table of Addition Facts – How Can Strategies Help?

+	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	12
3	3	4	5	6	7	8	9	10	11	12	13
4	4	5	6	7	8	9	10	11	12	13	14
5	5	6	7	8	9	10	11	12	13	14	15
6	6	7	8	9	10	11	12	13	14	15	16
7	7	8	9	10	11	12	13	14	15	16	17
8	8	9	10	11	12	13	14	15	16	17	18
9	9	10	11	12	13	14	15	16	17	18	19
10	10	11	12	13	14	15	16	17	18	19	20

Strategies and Properties:

- Add 0
- Add 1
- Add 2
- 10 as addend
- Make a 10
- Doubles
- Commutative Property
- Pretend a 10
- Build on known facts

Data from PISA about Fluency

- *PISA study of 250,000 15-year-olds: Students' use of memorization/rehearsal strategies are almost universally negatively associated with learning (OECD, 2015, 2010).*

The PISA logo consists of the letters 'PISA' in a bold, sans-serif font. The 'P' is red, the 'I' is blue, the 'S' is yellow, and the 'A' is green. To the left of the text is a stylized graphic of overlapping triangles in red, yellow, and grey.

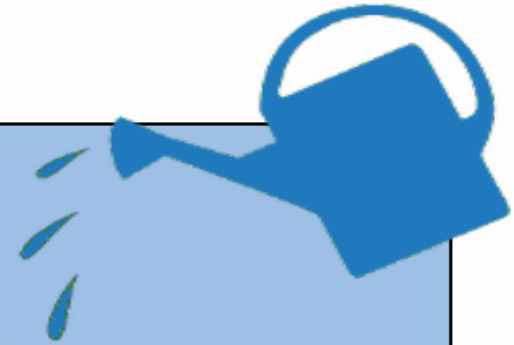
Programme for International Student Assessment

The OECD logo features a stylized globe icon to the left of the text 'OECD'. Below the text is the tagline 'BETTER POLICIES FOR BETTER LIVES' in a smaller font.

<https://www.oecd.org/pisa/>

Jennifer Bay-Williams Sept. 2021 CORE webinar *Harvesting Good Thinking: Growing Basic Fact Strategies to General Reasoning Strategies*

Build across Grade Levels



Basic Fact Reasoning Strategy: Making 10

General Reasoning Strategies:

→ **Make Tens** (Hundreds, etc.) [Whole numbers]

→ **Make a Whole** (Tenths, etc.) [Fractions and Decimals]

Whole Number Examples	Decimal Examples	Fraction Examples
$8 + 6 = 10 + 4 = 14$ $39 + 28 = 40 + 27$ $395 + 784 = 400 + 779$	$3.9 + 2.8 = 4.0 + 2.7$ $9.7 + 3.5 = 10.0 + 3.2$	$3\frac{3}{4} + 2\frac{1}{2} = 4 + 2\frac{1}{4} = 6\frac{1}{4}$ $\frac{2}{3} + \frac{11}{12} = \frac{8}{12} + \frac{11}{12}$ $= \frac{7}{12} + 1 = 1\frac{7}{12}$

Jennifer Bay-Williams Sept. 2021 CORE webinar *Harvesting Good Thinking: Growing Basic Fact Strategies to General Reasoning Strategies*

Fluency Activities with Number Sense in Mind

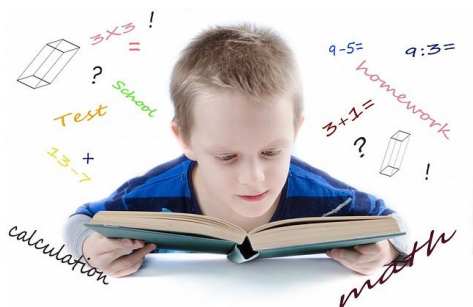


- Visual and verbal
- Paper and pencil
- Online

Verbal and Visual Activities



- Number Talks
- Counting Up and Counting Down
(Eureka Math calls this “Happy Counting”)
- Card Games



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Number Sense with Rational Numbers

Chat



Which is greater? How do you know?

$$S. \quad \frac{3}{4} - \frac{2}{3}$$

$$M. \quad \frac{3}{4} \cdot \frac{2}{3}$$

Adapted from *Open Up Resources*, Grade 6 Unit 4 Activity 16.1, 2021

Number Talks

A five- to fifteen-minute classroom conversation around purposefully crafted problems that are solved mentally.

Parrish and Dominick 2016

Counting Up and Down (“Happy Counting”)

- Count by twos starting at 8
- Count by twos again at 9
- Count by fives starting at 11

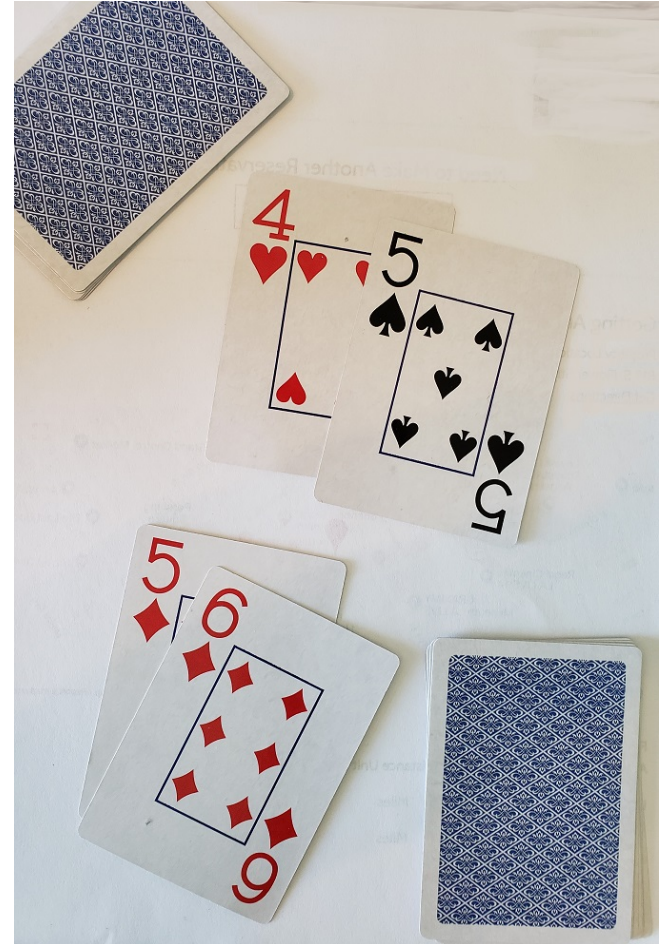
Advice:

- Start small
- Focus on where students are at to move forward
- Go back and forth across tens
- Discuss patterns and challenges
- Use a number line to model, display & build understanding
- Integers, fractions, decimals, exponents, algebraic expr.

Card Games

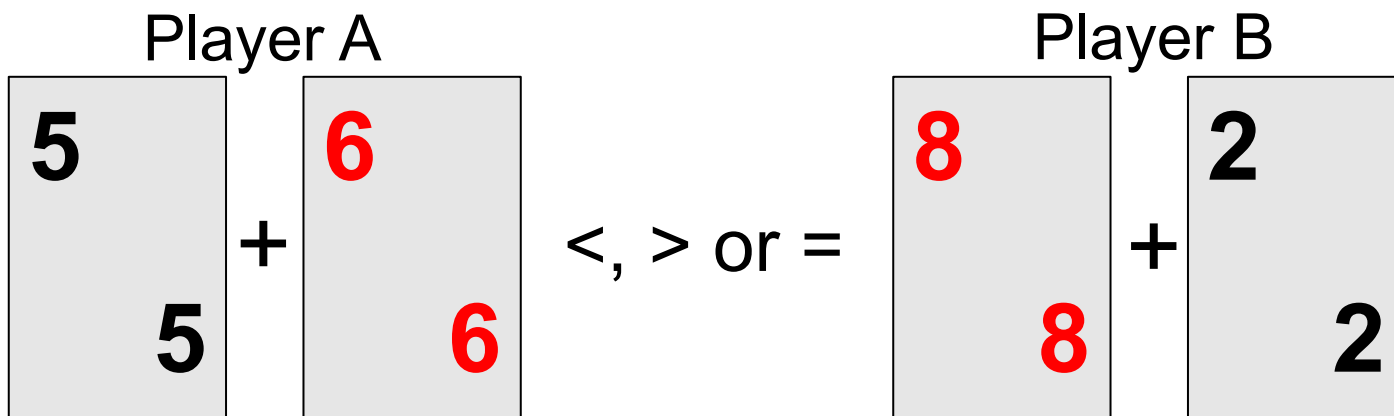
Examples

- Addition War
- Integer Addition War
- Fixed Factor War
- Fraction War
- Hit the Target



Whole Number or Integer War with Addition

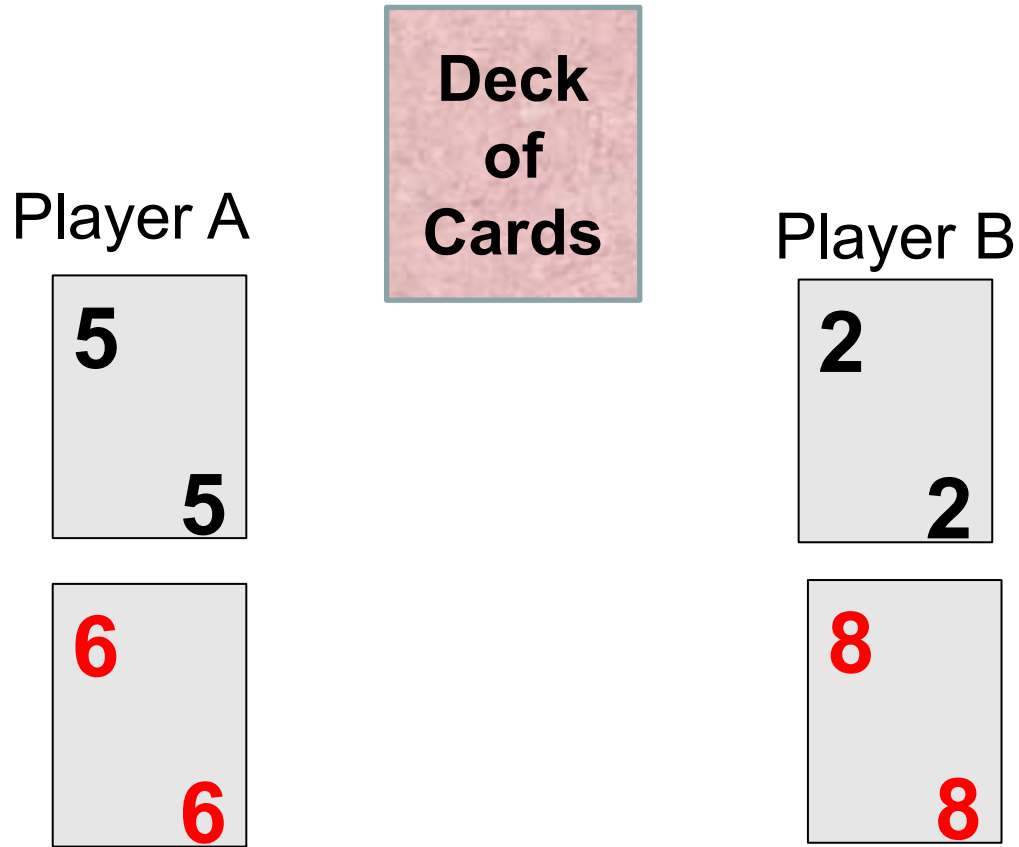
- Deck of cards (may remove face cards if desired)
- Integer War: Black suits = positive, red suits = negative
- Deal all cards out to two players, face down.
- Each player turns over their first two cards and determines which player has the greater sum of the two cards. The player with the greater sum wins the turn.



Whole Number: $5 + 6 <>= 8 + 2$

Integer: $5 + -6 <>= -8 + 2$

Fraction War



Who has the greater fraction?

Card War Game Options

Options:

- Include or don't include the face cards.
- Turn over three cards on each turn.
- Change operation to subtraction, multiplication or division.
- Students show work on worksheet and turn in worksheet (may be just some of the problems in order to keep from taking all the fun out of the game).

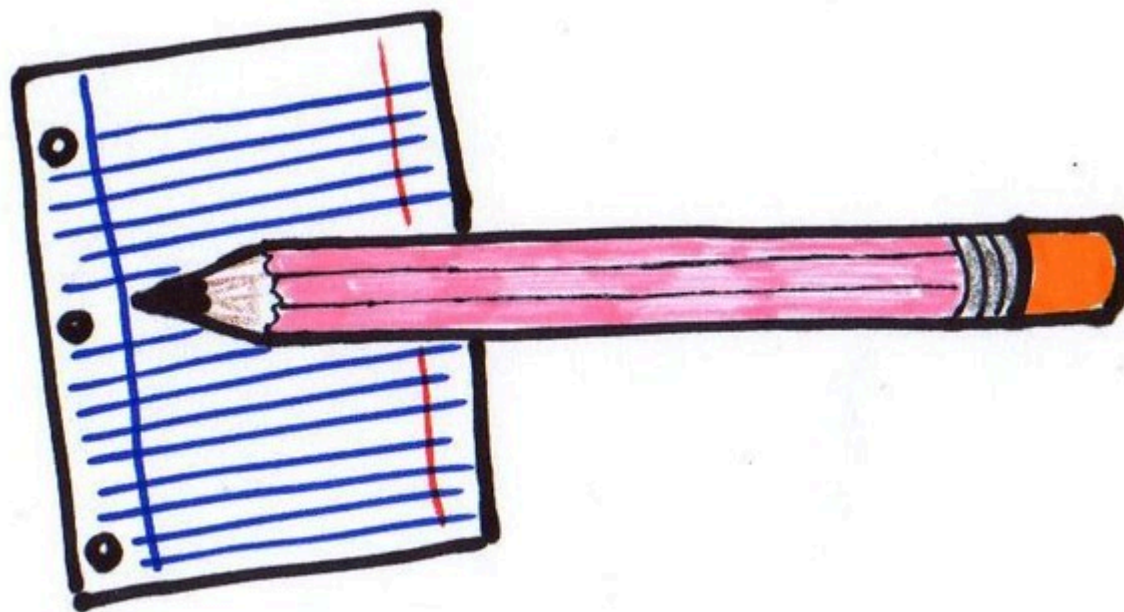
Hit the Target

- Target is 12

5 5	6 6	A A	8 8	2 2
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$5 + 6 + 1$	2×6	$2(5 + 1)$
$(8 - 6) \times 5 \times 1 + 2$		

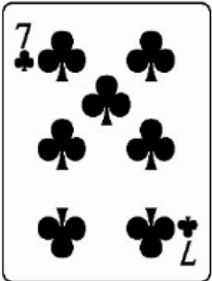
“Paper and Pencil” Activities



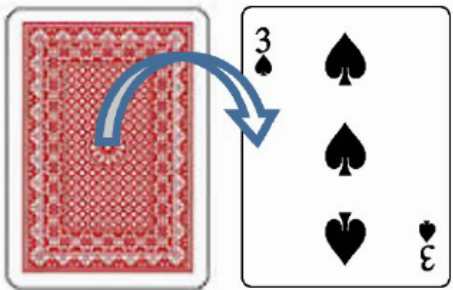
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Game: Fixed Factor War

Fixed Factor Card →
(Does not change.)

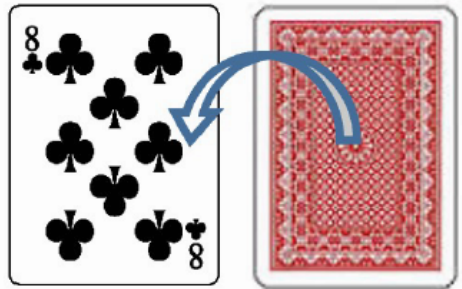


Player 1



“3 groups of 7 has a value of 21.
I know 2 groups of 7 is 14; so one
more group would be 21.”

Player 2



“8 groups of 7 is the same as 56.
I know that 7 groups of 7 is 49 and
1 more group of 7 would be 56.”

Bay-Williams on CORE Webinar (2021)

Game: Fill the Chart

Directions: Players take turns. On your turn, roll a 10-sided die (0 means 10). Multiply it by 10 or 9. Record the product in the empty space below that digit. Only record one product per roll.

FILL THE CHART

Directions: Players take turns. On your turn, roll a 10-sided die (0 means 10). Multiply it by 10 or 9. Record the product in the empty space below that digit. Only record one product per roll.

	1	2	3	4	5	6	7	8	9	10
$10 \times n$										
$9 \times n$										

Bay-Williams on CORE Webinar (2021)

IES Recommendation for Timed Activities

Recommendation 6 (based on “Strong” evidence – highest rating in terms of evidence from research)

Regularly include timed activities to build students’ retrieval of basic facts and fluent use of critical steps for more complex mathematics.

Institute of Education Sciences March 2021 publication, *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades Educator’s Practice Guide*

How to implement the Recommendation

Regularly include timed activities to build students' retrieval of basic facts and fluent use of critical steps for more complex mathematics.

1. **Identify already-learned topics** for activities to support fluency and create a timeline.
2. Choose the activity and accompanying materials to use in the timed activity and **set clear expectations**.
3. Ensure that students have an **efficient strategy** to use as they complete the timed activity.
4. Encourage and motivate students to **work hard** by having them chart their progress.
5. Provide **immediate feedback** by asking students to correct errors using an **efficient strategy**.

Obstacles	Advice
<p><i>“We do fluency worksheets every day, and my students are not improving.”</i></p>	<p>“... giving timed worksheets alone <i>does not</i> support fluency...Does the timed activity make sense for the intervention focus? Do your students have a way to find the answer if they don’t know it automatically? Are you giving students feedback in an immediate and meaningful way? Are students observing their own progress and setting goals?</p>
<p><i>“Some students seem to race through and guess.”</i></p>	<p>Remind students that accuracy is the goal; not how many problems were attempted. ... Suggest that they slow down and aim for accuracy next time to see if they can improve their score.</p>
<p><i>“Some of my students have anxiety when doing timed activities, especially when completing an activity with a large number of problems.”</i></p>	<p>If using worksheets, students may feel anxiety when seeing a large number of problems all at once. Make sure students know they are not expected to finish all of the problems and that there are more items on the worksheet than they are expected to complete ... Having students work as a group to “meet or beat” their previous collective score can also decrease the pressure they may feel if asked to perform individually.</p>

Individual White Board – Planful Practice

$$1. \quad \frac{1}{2} = \frac{\quad}{4}$$

$$5. \quad \frac{3}{4} = \frac{\quad}{20}$$

$$2. \quad \frac{1}{2} = \frac{\quad}{10}$$

$$6. \quad \frac{1}{4} = \frac{3}{\quad}$$

$$3. \quad \frac{1}{4} = \frac{\quad}{8}$$

$$7. \quad \frac{3}{9} = \frac{\quad}{3}$$

$$4. \quad \frac{1}{4} = \frac{\quad}{20}$$

$$8. \quad \frac{4}{12} = \frac{2}{\quad}$$

Individual White Board Practice - Recommendations

- Review and build on important knowledge
- Plan problems ahead of time
- Decide on how students will share
- Scaffold and adjust on the fly as needed
- Stop, reflect, model, teach, discuss as needed
- Limit time

Sprints – Planful Practice

- Practice of facts and skills
- Usually, 44 problems
- Version A and B (matching pairs of Sprints)
- Problems designed in quadrants
- Take advantage of the math built into the Sprint
 - **Practice + Understanding**

A

Number Correct: _____

Write the Missing Factor

1.	$10 = 5 \times \underline{\quad}$	
2.	$10 = 2 \times \underline{\quad}$	
3.	$8 = 4 \times \underline{\quad}$	
4.	$9 = 3 \times \underline{\quad}$	
5.	$6 = 2 \times \underline{\quad}$	
6.	$6 = 3 \times \underline{\quad}$	
7.	$12 = 6 \times \underline{\quad}$	
8.	$12 = 3 \times \underline{\quad}$	
9.	$12 = 4 \times \underline{\quad}$	
10.	$12 = 2 \times 2 \times \underline{\quad}$	
11.	$12 = 3 \times 2 \times \underline{\quad}$	
12.	$20 = 5 \times 2 \times \underline{\quad}$	
13.	$20 = 5 \times 2 \times \underline{\quad}$	
14.	$16 = 8 \times \underline{\quad}$	
15.	$16 = 4 \times 2 \times \underline{\quad}$	
16.	$24 = 8 \times \underline{\quad}$	
17.	$24 = 4 \times 2 \times \underline{\quad}$	
18.	$24 = 4 \times \underline{\quad} \times 2$	
19.	$24 = 3 \times 2 \times \underline{\quad}$	
20.	$24 = 3 \times \underline{\quad} \times 2$	
21.	$6 \times 4 = 8 \times \underline{\quad}$	
22.	$6 \times 4 = 4 \times 2 \times \underline{\quad}$	

23.	$28 = 7 \times \underline{\quad}$	
24.	$28 = 2 \times 2 \times \underline{\quad}$	
25.	$28 = 2 \times \underline{\quad} \times 2$	
26.	$28 = \underline{\quad} \times 2 \times 2$	
27.	$36 = 3 \times 3 \times \underline{\quad}$	
28.	$9 \times 4 = 3 \times 3 \times \underline{\quad}$	
29.	$9 \times 4 = 6 \times \underline{\quad}$	
30.	$9 \times 4 = 3 \times 2 \times \underline{\quad}$	
31.	$8 \times 6 = 4 \times \underline{\quad} \times 2$	
32.	$9 \times 9 = 3 \times \underline{\quad} \times 3$	
33.	$8 \times 8 = \underline{\quad} \times 8$	
34.	$7 \times 7 = \underline{\quad} \times 7$	
35.	$8 \times 3 = \underline{\quad} \times 6$	
36.	$16 \times 2 = \underline{\quad} \times 4$	
37.	$2 \times 18 = \underline{\quad} \times 9$	
38.	$28 \times 2 = \underline{\quad} \times 8$	
39.	$24 \times 3 = \underline{\quad} \times 9$	
40.	$6 \times 8 = \underline{\quad} \times 12$	
41.	$27 \times 3 = \underline{\quad} \times 9$	
42.	$12 \times 6 = \underline{\quad} \times 8$	
43.	$54 \times 2 = \underline{\quad} \times 12$	
44.	$9 \times 13 = \underline{\quad} \times 39$	

SPRINTS

It's not a race.

It's not a test.

It's practice.

It's learning strategies and building fluency.

Sprints – Recommendations

- **EMPHASIZE – NOT GRADED, NOT A TEST**
- **Vary delivery**
- **Match Sprints to student levels**
- **Focus on improvement**
- **Focus on understanding and practice**
- **Don't have to give Sprints A and B on same day**
- **Look over page 8 in handout for more ideas**

Online Resources

- **KenKen Puzzles**
- **Splats**
- ArcAdemics.com
- Sumaze (app for phone/tablets)
- Desmos.com



A second grade student in the Northshore School District, attends class remotely

KenKenPuzzle.com

3+ 1	2	3 3
7+ 3	1	3+ 2
2 2	3	1

7+ 4	4+ 3	1	3+ 2
3	6+ 2	4 4	1
1 1	4	5+ 2	7+ 3
3+ 2	1	3	4

Splats! From Steve Wyborney at <https://steveWyborney.com/>

The Fraction Splat! Series

By Steve Wyborney | March 8, 2017 | 38

Ad

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7

Splat!

Set 2.3

This set includes the numbers 12 - 20.

Splat!

Set 5.1

This set includes 2-Color Splats!
Remember: Splats that are different colors must be covering different numbers.

Accessing and Downloading Splats!

Steve Wyborney at steveWyborney.com

<https://steveWyborney.com/2017/02/splat/>

Choose Splat, click on Splat to download the PowerPoint deck (usually five decks per set)

- Do not need an account to access and download Splats



Final Reflection - Chat

What is your biggest take-away or what are you most excited about from this session??



Spend Some Time with 1 to 9

- Problem-solving and practice to build fluency and number sense. [wwwCorelearn.com](http://www.Corelearn.com)



Jennifer Bay-Williams' webinar with CORE, website, and books

<https://www.corelearn.com/webinar-harvesting-good-thinking/>



<http://kcm.nku.edu/mathfactfluency/index.php>

Feedback Link in Chat

Thank you!

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www.corelearn.com

